

## An Algebraic Approach to the Harmonic Oscillator Plus an Inverse Square Potential in Three Dimensions

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**Abstract:** The eigenfunctions and eigenvalues of the three-dimensional Schrödinger equation with a harmonic oscillator plus an inverse square interaction are obtained. A realization of the ladder operators for the wave functions is studied. It is found that these operators satisfy the commutation relations of an SU(1,1) group. The closed analytical expressions for the matrix elements of different

functions  $\rho$  and  $\rho \frac{d}{d\rho}$  with  $\rho = r^2$  are evaluated. Another hidden symmetry explores the relations

between the eigenvalues and eigenfunctions for substituting  $r \rightarrow ir$ .

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**Key words:** Inverse Square Interaction, Ladder Operators, SU (1, 1) Group, Matrix Elements

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### INTRODUCTION

It is well known that the algebraic method has been the subject of the interest in the wide variety of fields of physics. Systems displaying a dynamical symmetry can be treated by algebraic techniques [1, 2, 3, 4]. With the factorization method [5, 6, 7], we have established the ladder operators of a quantum system with some important potentials such as the Morse potential, the Pöschl-Teller one, the pseudoharmonic one, the infinitely square-well one and other quantum systems [8, 9, 10, 11, 12, 11, 12, 13, 14, 15, 16, 17, 18]. From those ladder operators we can finally constitute a suitable algebra and simultaneously obtain the matrix elements for some related functions. It should be addressed that our approach is different from the

traditional one, where an auxiliary variable was introduced [19], namely, we can construct the ladder operators only from the physical variable without introducing any auxiliary variable. Recently, the quantum system for the one-dimensional harmonic oscillator plus an inverse square interaction has been studied [20]. To our knowledge, however, the hidden symmetry of this quantum system is a gap to be filled in, which is the main purpose of the present work.

**The Eigenvalues and Eigenfunctions:** Now, we are studying the exact solutions, which are necessary to be used to construct the ladder operators and constitute a suitable algebra. Consider the Schrödinger equation with a potential  $V(r)$  that depends only on the distance  $r$  from the origin:

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi(r, \theta, \varphi) + V(r) \psi(r, \theta, \varphi) = E \psi(r, \theta, \varphi), \quad (1)$$

where  $\mu$  is the mass of the particle. In the present work, the potential  $V(r)$  is taken as the harmonic potential plus an inverse square interaction,

$$V(r) = \frac{1}{2} \mu \omega^2 r^2 + \frac{\hbar^2}{2\mu} \frac{\alpha}{r^2}, \quad (2)$$

where, the  $\omega$  is the frequency and  $\alpha$  is the strength of the external field.

For simplicity, the natural units  $\hbar = \mu = \omega = 1$  are employed throughout this paper if not explicitly stated otherwise. Owing to the symmetry of the potential, we take the wave functions with the form

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$$\psi(r, \theta, \varphi) = R(r)Y_{lm}(\theta, \varphi), \quad (3)$$

where, the radial wave function  $R(r)$  satisfies the following equation

$$\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[ 2(E - V(r)) - \frac{l(l+1)}{r^2} \right] R(r) = 0. \quad (4)$$

Defining the new variable  $\rho = r^2$ , equation (4) can be re-arranged as

$$\frac{d^2}{d\rho^2} R(\rho) + \frac{1}{\rho} \frac{d}{d\rho} R(\rho) - \left( \frac{1}{4} + \frac{\alpha + l(l+1)}{4\rho^2} - \frac{E}{2\rho} \right) R(\rho) = 0. \quad (5)$$

From the behaviors of the wave functions at the origin and at infinity, we can take the following ansatz for the wave functions

$$R(\rho) = \rho^s e^{-\frac{\rho}{2}} F(\rho), \quad (6)$$

with

$$s = \frac{-1/2 + \sqrt{\alpha + (l+1/2)^2}}{2}, \quad (7)$$

where, another solution  $s = (-1/2 - \sqrt{\alpha + (l+1/2)^2})/2$  is not an acceptable one in physics. Substitution of Eq. (6) into Eq. (5) allows us to obtain

$$\rho \frac{d^2}{d\rho^2} F(\rho) + (2s + 3/2 - \rho) \frac{d}{d\rho} F(\rho) + (E/2 - s - 3/4) F(\rho) = 0, \quad (8)$$

whose solution is nothing but the confluent hypergeometric solutions  ${}_1F_1(s - E/2 + 3/4, 2s + 3/2; \rho)$ . One can finally obtain the eigenfunctions as

$$R(\rho) = N \rho^s e^{-\frac{\rho}{2}} {}_1F_1(s - E/2 + 3/4, 2s + 3/2; \rho), \quad (9)$$

with a normalized factor  $N$  to be determined below:

From consideration of the finiteness of the solutions, it is shown from Eq. (7) that the general quantum condition is

$$s - E/2 + 3/4 = -n, \quad n = 0, 1, 2, \dots, \quad (10)$$

from which we have

$$E_n = 2(n + s + 3/4) = 3/2 + 2n + 2s, \quad (11)$$

which implies that the energy level is equidistant.

Recall that when  $n = E/2 - s - 3/4$  is a non-negative integer, the confluent hypergeometric functions can be expressed by the associated Laguerre polynomials [21]:

$$L_n^\alpha(x) = \frac{\Gamma(\alpha + n + 1)}{n! \Gamma(\alpha + 1)} {}_1F_1(-n, \alpha + 1; x),$$

from which, together with the following important formula [21]:

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{nm},$$

we can finally obtain the normalized radial wave functions

$$R(\rho) = N_n \rho^s e^{-\frac{\rho}{2}} L_n^{2s+1/2}(\rho), \tag{12}$$

with

$$N_n = \sqrt{\frac{2n!}{(n + 2s + 1/2)!}}. \tag{13}$$

**The construction of the ladder operators:** We address the problem of finding the creation and annihilation operators for the radial wave functions (12) with the factorization method. [8-18], the ladder operators can be constructed directly from the wave functions without introducing any auxiliary variable, namely, we intend to find differential operators  $\hat{L}_\pm$  with the following property

$$\hat{L}_\pm R_n(\rho) = l_\pm R_{n\pm 1}(\rho). \tag{14}$$

Specifically, we look for operators of the form

$$\hat{L}_\pm = A_\pm(\rho) \frac{d}{d\rho} + B_\pm(\rho), \tag{15}$$

where we stress that these operators only depend on the physical variable  $\rho$ .

To this end we start by establishing the action of the differential operator  $\frac{d}{d\rho}$  on the wave functions (12)

$$\frac{d}{d\rho} R_n(\rho) = \left( \frac{s}{\rho} - \frac{1}{2} \right) R_n(\rho) + N_n \rho^s e^{-\frac{\rho}{2}} \frac{d}{d\rho} L_n^{2s+1/2}(\rho). \tag{16}$$

One possible relation for the first derivative of the associated Laguerre functions is given in [21]

$$x \frac{d}{dx} L_n^\alpha(x) = n L_n^\alpha(x) - (n + \alpha) L_{n-1}^\alpha(x). \tag{17}$$

The substitution of this expression into (16) enables us to obtain the following relation

$$\left[ \frac{d}{d\rho} - \frac{n+s}{\rho} + \frac{1}{2} \right] R_n(\rho) = -\frac{n+2s+1/2}{\rho} \frac{N_n}{N_{n-1}} R_{n-1}(\rho). \tag{18}$$

Making use of Eq. (13) and introducing the number operator  $\hat{n}$  with the property

$$\hat{n} R_n(\rho) = n R_n(\rho), \tag{19}$$

we can define the following operator

$$\hat{L}_- = -\rho \frac{d}{d\rho} + s + \hat{n} - \frac{\rho}{2}, \tag{20}$$

with the following effect over the wave functions

$$\hat{L}_- R_n(\rho) = l_- R_{n-1}(\rho), \tag{21}$$

where,

$$l_- = \sqrt{n(n + 2s + 1/2)}. \tag{22}$$

As we can see, this operator annihilates the ground state  $R_0(\rho)$ , as expected from a step-down operator.

We now proceed to find the corresponding creation operator. Before proceeding to do so, we should make use of another relation between the associated Laguerre functions [21].

$$x \frac{d}{dx} L_n^\alpha(x) = (n+1)L_{n+1}^\alpha(x) - (n+\alpha+1-x)L_n^\alpha(x). \quad (23)$$

Substitution of this expression into Eq. (16) admits us to obtain

$$\left[ \frac{d}{d\rho} + \frac{n+s+3/2}{\rho} - \frac{1}{2} \right] R_n(\rho) = \frac{n+1}{\rho} \frac{N_n}{N_{n+1}} R_{n+1}(\rho). \quad (24)$$

Using Eq. (13) again, we can define the following operator

$$\hat{L}_+ = \rho \frac{d}{d\rho} + s + \hat{n} + 1 - \frac{\rho}{2}, \quad (25)$$

satisfying the equation

$$\hat{L}_+ R_n(\rho) = l_+ R_{n+1}(\rho), \quad (26)$$

with

$$l_+ = \sqrt{(n+1)(n+2s+3/2)}. \quad (27)$$

Therefore it is shown that the wave functions can be directly obtained from the creation operator  $\hat{L}_+$  acting on the ground state  $R_0(\rho)$ , namely,

$$R_n(\rho) = N_n \hat{L}_+^n R_0(\rho), \quad (28)$$

with

$$N_n = \sqrt{\frac{(2s+1/2)!}{n!(n+2s+1/2)!}}, \quad R_0(\rho) = \sqrt{\frac{2}{(2s+1/2)!}} \rho^s e^{-\frac{\rho}{2}}. \quad (29)$$

We now study the algebra associated to the operators  $\hat{L}_+$  and  $\hat{L}_-$ . Based on the results (21-22) and (26-27) we can calculate the commutator  $[\hat{L}_-, \hat{L}_+]$ :

$$[\hat{L}_-, \hat{L}_+] R_n(\rho) = 2l_0 R_n(\rho), \quad (30)$$

where we have introduced the eigenvalue

$$l_0 = \left( n + s + \frac{3}{4} \right). \quad (31)$$

We can thus define the operator

$$\hat{l}_0 = \left( \hat{n} + s + \frac{3}{4} \right). \quad (32)$$

The operators  $\hat{L}_\pm$  and  $\hat{L}_0$  thus satisfy the commutation relations

$$[\hat{L}_-, \hat{L}_+] = 2\hat{L}_0, \quad [\hat{L}_0, \hat{L}_-] = -\hat{L}_-, \quad [\hat{L}_0, \hat{L}_+] = \hat{L}_+, \quad (33)$$

which correspond to an SU(1, 1) algebra for the radial wave functions. The Casimir operator can be also expressed as

$$\hat{C}R_n(\rho) = [\hat{L}_0(\hat{L}_0 - 1) - \hat{L}_+\hat{L}_-]R_n(\rho) = (s^2 + s/2 - 3/16)R_n(\rho). \quad (34)$$

The hamiltonian  $\hat{H}$  can be expressed as:

$$\hat{H}|n\rangle = 2\hat{L}_0|n\rangle = (2n + 2s + 3/2)|n\rangle. \quad (35)$$

It is shown that there are four series of irreducible unitary representations for the SU(1, 1) algebra except for the identity representation [22]. They are the representation  $D^\pm(j)$  with a spectrum bounded below and above, respectively; the supplementary series  $D_s(Q, q_0)$  and the principle series  $D_p(Q, q_0)$ . Since the eigenvalues have the ground state, the representation of the dynamical group SU(1, 1) belongs to  $D^+(j)$ :

$$\begin{aligned} I_0|j, v\rangle &= v|j, v\rangle, \\ I_-|j, v\rangle &= [(v+j)(v-j-1)]^{1/2}|j, v-1\rangle, \\ I_+|j, v-1\rangle &= [(v+j)(v-j-1)]^{1/2}|j, v\rangle, \\ v &= -j+n, \quad n = 0, 1, 2, \dots, \quad j < 0. \end{aligned} \quad (36)$$

In comparison with Eqs. (21-22), (29-30) and (33-34) we have  $j = -(s + 3/4)$ ,  $v = n + s + 3/4$  and  $R_n(\rho) = |j, v\rangle$ .

On the other hand, the following expressions in terms of the creation and annihilation operators  $\hat{L}_\pm$  and  $\hat{L}_0$  can be obtained as

$$\rho = 2\hat{L}_0 - \hat{L}_- - \hat{L}_+, \quad (37a)$$

and

$$\rho \frac{d}{d\rho} = \frac{1}{2}(\hat{L}_+ - \hat{L}_- - 3/2). \quad (37b)$$

The matrix elements of these two functions can be analytically obtained in terms of Eqs. (21), (31) and (34) as

$$\begin{aligned} \langle R_m(\rho) | \rho | R_n(\rho) \rangle &\equiv \int_0^\infty R_m(r) r^2 R_n(r) dr \\ &= (2n + 2s + 3/2) \delta_{m,n} \\ &\quad - \sqrt{n(n + 2s + 1/2)} \delta_{m,(n-1)} \\ &\quad - \sqrt{(n+1)(n + 2s + 3/2)} \delta_{m,(n+1)} \end{aligned} \quad (38a)$$

and

$$\begin{aligned}
 \langle R_m(\rho) | \rho \frac{d}{d\rho} | R_n(\rho) \rangle &\equiv \int_0^\infty R_m(r) \frac{r}{2} \frac{d}{dr} R_n(r) dr \\
 &= \frac{1}{2} \sqrt{(n+1)(n+2s+3/2)} \delta_{m,(n+1)} \\
 &\quad - \frac{1}{2} \sqrt{n(n+2s+1/2)} \delta_{m,(n-1)} \\
 &\quad - \frac{3}{4} \delta_{m,n}
 \end{aligned}
 \tag{38b}$$

where, the integral range  $r \in (0, \infty)$ . It is shown that this is a very simple method to calculate the matrix elements from these ladder operators.

Before ending this section, it should be pointed out that there exists another hidden symmetry for this quantum system. As discussed in our previous work [23], it is found that the eigenfunctions (12) (the variable  $\rho = r^2$ ) vanish not only as  $r \rightarrow \infty$ , but also as  $r \rightarrow i\infty$ . The substitution  $r \rightarrow ir$ , also demonstrates intimate connections between the energies, that is to say, the eigenvalues  $E$  of quantum system will change to  $-E$  but keeping other parameters unchanged.

### CONCLUSION

In this paper we have studied the eigenvalues and eigenfunctions for the one-dimensional harmonic oscillator plus an inverse square interaction and then established the creation and annihilation operators directly from the eigenfunctions (12) with the factorization method. We have derived a realization of dynamic group only in terms of the physical variable  $\rho$ , without introducing any auxiliary variable. It is shown that these operators satisfy an  $SU(1, 1)$  dynamic group. The representation of the bound states of this quantum system is described by the representation  $D^+(j)$  with a spectrum bound below. The matrix elements of the different functions  $\rho$  and  $\rho \frac{d}{d\rho}$  with  $\rho = r^2$  are also analytically obtained from the ladder operators  $\hat{L}_\pm$  and  $\hat{L}_0$ . This method can be generalized to other wavefunctions and represents a simple and elegant approach to obtain these matrix elements.

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