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Construction and Application of a Statistical Test for Coefficient of Variation on Normal Distributions

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Abstract: In this study a novel statistical test is derived for the Coefficient of Variation (CV) under normal distributions. This is a newly derived test with value to engineering sciences in aspects of production of accurate items. The CV can measure the precision of a measuring instrument, among other applications. In order to determine instrument reliability, start by generating measures using the instrument. The CV is then calculated to determine if the measures generated by the instrument are concentrated around a central point. In use of normal distribution presumption, or approximation, applicable properties of the normal distributions lead to involvement of the chi-square and t-distributions. A CV test is then constructed, and two illustrative examples conclude the discussion.

Keywords: Central Limit Theorem, Coefficient of Variation, Law of Large Numbers, Normal Distribution

Introduction

Background and Definition of the Problem

Let X be a continuous random variable with mean μ and variance σ^2. As it is usually difficult to know these parameters, sample values are often used to estimate them (Hinkelmann and Kempthorne 2008; Nicholas 2006). Consider a random sample of n observations from X denoted by $X_1, X_2, \ldots, X_n$. Let $\bar{X}$ and S^2 be the sample mean and sample variance. These sample estimates are respective estimators of μ and σ^2. The square root σ of σ^2 is the population standard deviation. Similarly, the square root S of S^2 is the sample standard deviation.

Review of Existing Literature

Several authors (Bennett and Briggs 2005; Gelman 2005; 2008; Kleijnen 2000; Reed *et al*. 2002) define the population CV as:

$$V = \frac{\sigma}{\mu} \tag{1}$$

According to several authors (Armitage 2005; Kleijnen and Sargent 2000; Reed *et al*. 2002), the CV value provides the precision of any measuring instrument or sampling procedure used. Kleijnen (2000) remarks further that there is no known exact statistical test for testing the coefficient of variation. This remark stimulated interest in developing such a test.

Purpose of the Present Study

This paper aims to contribute in testing hypotheses that involve V.

Consider the gamma function which has the form $\Gamma(k) = \int y^{k-1} e^{-y} dy, k > 0$. Before the discussion takes off, two assumptions are made. First, it is assumed that the sample sizes used in this study are large enough to offset the pitfall of lack of representativeness that could result from some small samples. The second assumption is that samples are drawn from populations that are normally distributed.

Formulation of the Hypothesis

Suppose that experience shows that X has a specified mean $\mu_0 (\neq 0)$ and variance $\sigma_0^2 (> 0)$. Then it is tempting to believe that the general variation in repeated measurements taken from X is given by the CV:

$$v_0 = \frac{\sigma_0}{\mu_0} \tag{2}$$

To test the assumption that the CV is v_0, let a random sample of size n from X be given. Also, let $\bar{X}$ and S^2 be the respective sample estimates of μ_0 and σ_0^2. The aim is to test the null hypothesis H_0 given by:

$$H_0: v = v_0 \tag{3}$$

Denote by H_a, any alternative of H_0. The test statistic V is an estimate of v, where:

$$V = \frac{s}{\bar{x}} \tag{4}$$

Construction of a Hypothesis Test for V

A chi-square distribution derives from the quotient of a standard normal and chi-square distribution. Consider two independent random variables, Z which is standard normal random in distribution and X_k^2, which has a chi-square distributed random variable with k degrees of freedom. Pillai (2016) and Westfall (2013) showed that the random variable t defined in Equation (5) below, has a t-distribution with k degrees of freedom, where:

$$t = \frac{Z}{\sqrt{\frac{\chi_k^2}{k}}} \tag{5}$$

This section uses sampling distributions of known statistics that are associated with V to derive a statistical test given by equation (4). Cox (2006) demonstrates that for any sample size n from a normal distribution, the quantity:

$$\frac{(n-1)S^2}{\sigma_0^2} \tag{6}$$

has a χ_{n-1}^2 distribution $\bar{X}$ and S^2 are independent random variables. By implication, $\bar{X}$ and S are independent random variables.

In the forthcoming discussion, the symbol '~' shall denote the phrase 'is distributed as'. Recall the random sample $X_1, X_{2,\ldots\ldots,}X_n$ from X with mean μ_0. Define:

$$X_i^* = X_i - \mu_0 \ \ for\ i = 1,2,\ldots,n$$

Therefore, the mean of X_i^* is $\mu_i^* = 0$ and variance is $\sigma_i^{2*} = \sigma_0^2$ for all $i = 1,2,\ldots,n$.

Tabachnick and Fidell (2007) ascertain that if the sample was drawn from a normal distribution, then for any n,

$$\sqrt{n\frac{\bar{x}^*}{\sigma_0^*}} \tag{7}$$

has an approximate standard normal distribution. Then using equations (5) and (6), the statistic:

$$\frac{\sqrt{n\frac{\bar{X}^*}{\sigma_0^2}}}{\sqrt{\frac{(n-1)S^2}{\sigma_0^2}/(n-1)}} = \frac{\sqrt{n\bar{X}^*}}{S} \sim \ t_{n-1} \tag{8}$$

where, t_{n-1} is the t-distribution with n-1 degrees of freedom.

From expansion of (8):

$$\frac{\sqrt{n\bar{X}^*}}{S} = \frac{\sqrt{n\bar{X} - n\mu_0}}{S}$$

Then applying the above equation on (8) leads to:

$$\sqrt{\frac{n\bar{X} - n\mu_0}{S}} \sim \ t_{n-1} \tag{9}$$

Recalling Equation (4):

$$V = \frac{s}{\bar{X}}$$

Then, in equation (9) we will be able to conclude and use the equation:

$$\bar{X} = \frac{S}{V}$$

Add-on operations from Equations (8) and (9) are:

$$\sqrt{n\frac{S}{V} - n\mu_0} \sim S t_{n-1}$$

$$\Rightarrow n\frac{S}{V} - n\mu_0 \sim S^2 t_{n-1}^2$$

$$\Rightarrow n\frac{S}{V} \sim n\mu_0 + S^2 t_{n-1}^2$$

Variate relationships of inverses and transformation of random variables exist with the basic mathematical operations (Forbes *et al*. 2000). Assuming that these mathematical properties, coupled with the aforesaid manipulations, are permissible with ~, then from equation (9):

$$V \sim \frac{n}{S\left(n\mu_0 + St_{n-1}^2\right)} \tag{10}$$

The calculated value of V is denoted by V_{calc}, while $t_{m,p}$ shall denote the table value from t-distribution with m degrees of freedom and the probability p on the upper region of the table of t-distributions.

The next discussions use capital letters for denoting the estimators and small letters for denoting estimates. Estimators are statistics and therefore random variables with probability distribution. Small letters are used for calculating sample estimates from actual data observed.

Theoretical Construction of Critical Region

This section is intended to construct the critical region for the test initiated at Equation (3). Consider a random sample for the normal distribution with mean μ. The next formation uses equation (5), which in brief, states the fact that if Z, which is distributed as standard normal (denoted by $Z \sim N\ (0,\ 1)$) and $Y \sim X_n^2$ are independent random variables (Broverman, 2001), then:

$$T = \frac{Z}{\sqrt{Y/n}} \sim t_{n-1}$$

Research statisticians (Bless and Kathuria 1993; Wackerly *et al*. 2007) show that the *t*-test is as follows.

From Equation (10), the test statistic:

$$T = \sqrt{n\frac{S}{V} - n\mu_0} \sim St_{n-1}$$

The test procedure follows. The null hypothesis is:

$$H_0 : V = v_0 \tag{11}$$

Upper Alternative

Suppose that the alternative hypothesis is:

$$H_a : V > v_0 \tag{12}$$

Then, by virtue of *V* being indirectly proportional with μ_0, we reject H_0 if:

$$V_{calc} < \frac{n}{S\left(n\mu_0 + S^2 t_{n-1,\alpha}^2\right)} \tag{13}$$

where, *s* is the estimate of the population standard deviation.

Lower Alternative

Let:

$$H_a : V < v_0 \tag{14}$$

Then, reject H_0 if:

$$V_{calc} > \frac{n}{S\left(n\mu_0 + S^2 t_{n-1,\alpha}^2\right)} \tag{15}$$

Two-Sided Alternative

Lastly, let:

$$H_a : V \neq v_0 \tag{16}$$

Then, reject H_0 if:

$$V_{calc} < \frac{n}{S\left(n\mu_0 + S^2 t_{n-1,\alpha}^2\right)} \tag{17}$$

Or:

$$V_{calc} > \frac{n}{S\left(n\mu_0 + S^2 t_{n-1,\alpha}^2\right)} \tag{18}$$

Methodology

This section basically explains the test procedure. The test approach for the CV is guided by the following premises:

$$H_0 : V = v_0$$
$$Vs$$
$$H_a : Not\ H_0$$

In this case, where v_0 is specified, but neither μ nor σ are stated, then for the test statistic *V*, μ_0 shall be estimated by using:

$$\mu_0 = \frac{s}{v_0} \tag{19}$$

Also at Equation (10), for sufficiently large sample size *n*, z_p will be employed in the place of $t_{n-1,p}$ due to use of the approximate normal distribution from the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) (Berenson *et al*. 2012; Fischer 2011; Freedman 2005; Kleijnen 2000; Reed *et al*. 2002). In fact, from LLN and CLT, for large n (or degrees of freedom), the distribution of the t-statistic approaches a standard normal distribution.

Pragmatic Resolution on Critical Regions

Deficiency of the t-Test

One probable problem in the *t*-test is that the investigator can determine the significance level. Thus, the user can retain or reject the null hypothesis based upon their decision (Mankiewicz 2000). Even though not proper, an analyst can potentially influence the confidence interval in order to achieve the preferred result.

Another concern with the t-test is that the results are precisely truthful only with normal populations (Raju 2005). On the other hand, Rice (2006) counsels that real populations are never exactly normal. The techniques for t-tests are fortunately reasonably robust against non-normality in the population except in the presence of outliers or significant skewness. Larger samples greatly improve the accuracy of the critical values in *t*-distributions when the population is not normal. Dodge (2008) explains that due to the CLT, the sampling distribution of the sample mean from a sufficiently large sample is approximately normally distributed. Also, as the sample size increases, the sample standard deviation converges to the population standard deviation. Sawilowsky (2005) warns analysts to be cautious when applying the t-tests because of hidden influences from influential data. Fay and Proschan (2010) also suggest alternative tests when conditions for use of a t-test are questionable, or the test is likely going to be subjected to misleading effects. In the case of the test in this paper, a squared t has emerged and there are therefore doubts in the direct use of the t-tests. Thus, the t-test is proposed for application only to as far as it relates to other distribution(s).

Approximate Construction

Let Z_1, Z_2,...,Z_k be normal, identically and independently distributed (iid) with mean 0 and variance 1, denoted by $Z_i \sim$ iid $N(0,1)$ for $i = 1,2,\ldots,k$. Then, Bagdonavicius *et al*. (2011) and Lomax (2007) show that:

$$X=Z_1^2+Z_2^2+\ldots+Z_k^2 \sim \chi_k^2 \tag{20}$$

This equation is fundamental in relationships of the normal *t*-, X^2 and F-distributions (Corder and Foreman, 2014; Jaynes, 2003).

Relationship between X^2 and F-Distributions

For *n* independent observations from a $N(\mu, \sigma^2)$, the sum of the squared standard scores has a chi-square distribution with *n* degrees of freedom (Bagdonavicius and Nikulin 2011). Chi-squared distribution only depends on degrees of freedom, which in turn depends on sample size *n*. The standard scores are computed using population μ and σ^2. However, the actual values of μ and σ^2 are usually not known. When μ and σ^2 are estimated from the sampled data, the degrees of freedom are less than *n*.

The *F* distribution is used in tests to compare if two variances are equal (DeGroot and Schervish 2011). The test starts with two independent populations, Y_1 and Y_2, each being normally distributed and having equal variances. Then, let $Y_1 \sim iid\ N(\mu_1, \sigma^2)$ and $Y_2 \sim iid\ N(\mu_2, \sigma^2)$, and draw two independent random samples fromeach population, with sample sizes n_1 from population 1 and n_2 from population 2.

According to Larsen and Marx (2001), in constructing the *F* distribution using data from each of the two samples, estimate σ^2 using the pooled variance from sample variances s_1^2 and s_2^2. Both S_1^2 and S_2^2 are random variables. Furthermore, their ratio is a random variable:

$$F=\frac{estimate\ \ of\,\sigma_1^2}{estimate\ \ of\,\sigma_2^2}=\frac{s_1^2}{s_2^2} \tag{21}$$

This can further be presented mathematically and thus:

$$F=\frac{\chi_{n_1-1}^2/(n_1-1)}{\chi_{n_2-1}^2/(n_2-1)} \tag{22}$$

Therefore, the random variable *F* defined from two independent chi-square variables has an F-distribution.

Relationship between t- and F-Distributions

The procedures described are mathematically valid, as they satisfy mathematical principles. However, reality, according to Abramowitz and Stegun (2012), is that $t_k^2 = F_{1,k}$. This means that the squared t-distribution with *k* degrees of freedom is the same as the F-distribution with 1 and *k* degrees of freedom.

F-Test

An *F*-test is a statistical test with the test statistic that has an F-distribution when the null hypothesis is true (Maddala and Lahiri 2009). It is used to compare statistical models that have been fitted to a dataset to identify the model with best fit to the study population. As shown, the F-distribution is a ratio of two independent chi-square random variables that are divided by the respective degrees of freedom.

The F-test is developed to test equality of two population variances (Bulmer 2012). It does this by comparing the ratio of two variances. Thus, if the variances are equal, the ratio of the variances equals 1. If the null hypothesis is true, then the F-test statistic can be simplified. The ratio of sample variances is the test statistic used. If the null hypothesis is false, then the decision is to reject the null hypothesis that the ratio is equal to 1 and thus the assumption of equality.

The F-tables only give levels of significance for right tailed tests. Since the F-distribution is not symmetric, and there are no negative values, we do not simply take the opposite of the right critical value to obtain the left critical value (Carlberg 2014). The way to find a left critical value is to reverse the degrees of freedom, find the right critical value, and then take the reciprocal of this value.

Formation of F-Test

The *F* is formed by chi-square (Sawilowsky 2002) and therefore many of the chi-square properties carry over to the F-distribution. In conducting the F-tests, then the following conditions are valid (Ryabko *et al*. 2004): The F-values are non-negative. The F-distribution is non-symmetric. Its mean is about 1. There are two independent degrees of freedom, where one is for the numerator and the other is for the denominator. Also, there are many different F-distributions, one for each pair of degrees of freedom.

The approach in using the F-distribution is to avoid left critical values (Bagdonavicius and Nikulin 2002). Generally, the left critical values are difficult to calculate. As a result they are often avoided. A strategy is to influence the F-test towards a right tailed test by assigning the sample with the large variance in the numerator and the smaller variance in the denominator. Even though it does not matter which sample has the larger sample size, it matters that the sample having the larger variance is placed in the numerator.

In developing the F-tests, the following assumptions are necessary in the use of the F-distribution (Cacoullos 1965; Fadem 2012). The larger variance should always be placed in the numerator. The test statistic is:

$$F = s_1^2 / s_2^2 \quad where \quad s_1^2 > s_2^2$$

In the case where a two-sided alternative hypothesis would have been appropriate, the approach should be to divide the significance level (or alpha) by 2 and then obtain the right critical value (Galecki and Burzykowski 2013). When the degrees of freedom are not given in the table, then the value with the larger critical value should be chosen. This is the smaller degrees of freedom and reduces the likelihood of type I error.

The study populations providing the samples have to be normal (or approximately normal at least) and the sampling must be conducted using a random sampling method. These random samples must be independent.

Numerical Illustrations

Car Crash Prices Data

The following scores in Table 1 are the prices in ZAR1000's of accident damaged cars of the same model at Denver Company, Johannesburg (The Star Classified 1998:4). Assume that before the accidents, all these cars were of equal value.

Suppose further that if prices of these cars are identical, the CV is expected to be at most 10%. The question to be addressed is whether the damages caused by the accidents on these cars are significantly dissimilar.

Table 1: Prices of damaged cars at Denver Co

7.5	8	11	13	14	9.75	19.5	7	13	9.5	17.5	9
15	8	14	8.5	12	9	7	8	8.5	12.5	7	4.5
6	7.5	8.5	13.5	15	24	14	12	29			

In addressing this question, the information given could be used to determine if CV of at most 0.1 is an acceptable assumption. The answer is deduced in the following set of operations:

(1) $H_0 : V \leq 0.1 \;\; vs \;\; H_0 : V > 0.1$

(2) $Level\ of\ significance, \alpha = 0.1$

(3) $V_{calc} = \frac{s}{\bar{x}} = \frac{5.217408759784}{11.583333333333} = 0.4504$

(4) From equation (15), reject H_0 if:

$$V_{calc} < V_{crit} = \frac{n}{S\left(n\mu_0 + S^2 t_{n-1,\alpha}^2\right)}.$$

Now, $t_{n-1,\alpha} = t_{32,0.1} = 1.3086, \frac{\mu_0}{\sigma_0} = \frac{1}{V} = 10$ and n = 33. Then the test statistic is:

$$V_{crit} = \frac{n}{S(n\mu_0 + S^2 t^2{}_{n-1,\alpha})}$$
$$= \frac{33}{5.2174(33 \times 10 \times 5.2174 + 5.2174^2 \times 1.3086^2)}$$
$$= 0.0036$$

Since V_{calc}>V, then H_0 should not be rejected at the 10% significance level. It can be decided that the variance of the prices of these cars is not significantly large compared to the mean. Therefore the damages to the cars were not expressively different.

Examination Data

Howell (2002) provides a possibility to test under the following conditions. Consider a teaching professor in the Department of Psychology at a known University who teaches a group of students with learning disabilities. She finds it rewarding if after each test has been written she can determine if the test was easy or difficult. According to her, this has positive implications for her teaching approach of the subject as well as for assessment purposes. One of her discoveries is that if the student marks are in the ratio of standard deviation to mean of 3:10 or more, it is impossible to determine the level of examination difficulty. Recently, she examined her class of 700 students. The marks (in percentages) produced an average of 53 with a standard deviation of 14.143. The analysis should establish if the professor's teaching can benefit from the latest test.

In response to the above question, it is known that the professor can benefit upon determining if the test was easy or difficult. The hypothesis that the ratio of standard deviation to mean is 3:10 or more is made, which is to assume that she will not benefit from the test. A formal test of hypothesis is made. The professor cannot say if the test was easy or difficult if:

$$H_0 : V \geq 0.3$$

This is tested against the alternative hypothesis that:

$$H_a : V < 0.3$$

The alternative hypothesis implies that the professor can determine if the test was easy or difficult.

Given that n = 700, $\bar{x} = 53$ and s = 14.143, then the test is as follows:

(1) $H_0 : V \geq 0.3 \; vs \; H_a : V < 0.3$
(2) $Choose \;\; a = 0.05$
(3) $V_{calc} = \frac{s}{\bar{x}} = \frac{14.143}{53} = 0.26685$
(4) From (17), reject H_0 if:

$$V_{calc} > V_{crit} = \frac{n}{S\left(n\mu_0 + S^2 t^2_{n-1,\alpha}\right)}.$$

Now, $t_{n-1,\alpha} \cong z_{0.05} = 1.645, \frac{\mu_0}{s} = \frac{1}{v_0} = \frac{10}{3}$, the test statistic is:

$$V_{crit} = \frac{n}{S\left(n\mu_0 + S^2 t^2_{n-1,\alpha}\right)}$$
$$= \frac{700}{14.143\left(700 \times (10/3) \times 14.143 + 14.143^2 \times 1.645^2\right)} = 0.0015.$$

Since $V_{calc} > V_{crit}$, then H_0 should be rejected. It seems that the level of difficulty of the test can be ascertained. It is therefore acceptable to believe that at the 5% significance level, the professor can benefit from this test for her teaching, or from the information available.

Conclusion

The assertion by Kleijnen and other researchers that there are no known exact statistical tests for the CV stimulated this paper. Auspiciously, the gap identified has been addressed with the work in this paper. The paper contributes to knowledge by increasing the tools for analysis, which in this study is a statistical test for CV under normal distributions. These tests also envelop approximate normality, and extend to the CLT and LLN.

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Authors' Contributions

Gezani Richman Miyambu: Miyambu provided the mathematical methods for the test statistics and wrote some sections of the paper.

Solly Matshonisa Seeletse: Seeletse selected the case studies used in the numerical illustrations and performed the calculations.

Ethics

The study benefitted from secondary data and the necessary acknowledgements have been provided by means of references.

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