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Covering Approximations Approach to Interval Ordered Information Systems

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Abstract: It is been proved that the theory of rough set is very beneficial in working with conflict problems induced by the information endearment. The original idea of the rough set is not accurate, however, when preference orders of characteristics domains (standard) are to be considered. Seeking the issue of covering approximation relative to a control relation in interval-ordered information systems is a new mathematical tool to set up a control-based rough set approach. These Mathematical tools apply some topological generalizations, and control relations, covering approximations, and interval information systems. These results using our approach are more accurate than using a classical approach such as the approach of Pawlak. Topologies reproduce types of information systems with one value. By defining a control relation to interval information systems, we generalized the Pawlak approximation space to a covering approximation space and then we involve this approach to work with the interval-ordered information systems. Results gained by the proposed approach to generate two different rough approximations called *j*-lower and *j*-upper approximations. We applied *j*-rough notions such as *j*-rough membership, *j*-rough equality, and *j*-rough inclusion relations using topological generalizations. Our proposed method has improved the results gained from interval information systems. In this Research, there are many improved approaches to investing generalized approximation relative to a control relation in interval information systems, and we use the covering approximation in the rough set approach. This is a generalization of the Pawlak approach applied to interval information systems. This approach opens the way for other generalizations by making new algorithms, which simplify the calculations on it, and we can add more topological concepts to support decision-making in real-life applications.

Keywords: Topological Spaces, Rough Sets, Rough Approximations, Accuracy Measures, Data Classifications

Introduction

We addressed in this study how to generalize classification approximation spaces to new spaces using topology. The research problem here is how to take an accurate decision from the available data of ordered information systems. The process of solving this problem count on the ability to assort objects of this system using topological tools and making use of applications. Tools used in this approach are linked with the selection of topologies (Salama, 2018a; El Barbary and Salama, 2018; Abu-Gdairi *et al*., 2022). This view of classification and approximations is near to the theory of abstract topological spaces. This view is based on a topological structure that consists of a class of subsets of the universe classified according to its satisfaction with some axioms. Accordingly, the topological spaces used in this approach are not metric spaces for most types of information representations.

There are many methods and tools for studying classification problems using data in different types of information systems. The theory of rough set is a kind of recent method which can be generalized to topological tools. This theory depends on a particular kind of relations called equivalence relations that generalized a special class of topological spaces known as quasi-discrete topological spaces. This theory concerns knowledge representation that assumes objects under study as functions, which maps these objects into the corresponding value sets (Al-shami *et al*., 2021a; El-Bably and Al-Shami, 2021; Al-shami *et al*., 2021b)

Previous Studies

Pawlak (1982) Pawlak published his first paper on the theory of rough set, which is a mathematical key for decision-making and knowledge discovery. Many researchers around the world contributed fundamentally to its improvement and use it in different applications. In a simple period, many extensions of this theory appeared online using fuzzy sets (Chakrabarty *et al*., 2002; Yang and Hinde, 2010). Other novel approaches to fuzzy sets depending on fuzzy covering and rapprochement of different kinds of rough sets introduced by covering are addressed by (Deng *et al*. 2007; Liu and Sai, 2009). The concept of a generalized fuzzy rough set and the measures of it depending on fuzzy covering appeared and studied by (Li *et al*. 2008; Chen *et al*., 2008). The study took another turn when the researchers directed to change the type of relations used from its foundation to use tolerance relations to generalize rough sets in (Ouyang *et al*., 2010) and use similarity relations (Slowinski and Vanderpooten, 2000). Researchers specializing in topology have used this theory from the point of view of topological concepts to generalize it on digital topology (Abo-Tabl, 2014) and topological spaces (Lashin *et al*., 2005). In addition, the properties of fuzzy rough sets are sought from a topological view (Qin and Pei, 2005). Topologists have produced much research to generalize and solve many problems using rough sets to complete and incomplete information systems (Salama, 2010). When you drop some conditions of the equivalence relation you can use the neighborhood approach to information retrieval as done by (El Barbary *et al*. 2018). Topologists studied several interesting and meaningful generalizations and applications (Salama, 2016; Salama, 2008b) to rough sets using topology such as Feature selection for document classification (El Barbary and Salama, 2019), retrieve-missing values in incomplete information systems (Salama and El-Barbary, 2017). Different approaches have been achieved using topologies such as sequences of topological near open and near closed sets (Salama, 2020a), bitopological approximation space (Salama, 2020b) and approaches for rough continuous functions (Salama *et al*., 2021). (Abu-Donia, 2008) Abu-Doina achieved a new comparison between different kinds of approximations by using a family of binary relations without any conditions. The approaches that tried to generalize rough sets tried to change some properties of binary relations such as those that appear in (Zhang *et al*., 2009; Zhu, 2009; Zhu, 2007). A model of a human heart via graph Nano topological spaces is a real-life application of rough sets using topology (Nawar *et al*., 2019).

It is noteworthy that rough sets play a critical part in decision-making in an information system via new topology (Paul, 2016), approach to incomplete information systems (Kryszkiewicz, 1998), soft rough approximation (Feng, 2011), and also Naveed (Yaqoob and Aslam, 2012) found a notion of generalized rough set.

Philosophy of Generalized Rough Sets

The beautiful feature of the theory of rough set, which is the basis of its philosophy and its widespread, is that it does not use many tools and depends entirely on the available data. The basic tool of this theory is considered to be the approximation space. It is a couple $A = (U, R)$, such that U is a universe set and R is an equivalence relation that is the generator of the approximation space. $[y]_{\Re} \subseteq U$, is called the elementary set of this theory that is used in defining rough approximations. The rough lower and rough upper approximations of $Y \subseteq U$ are known by $\underline{\Re}(Y) = \bigcup\{\lceil Y \rceil_{\Re} : [y]_{\Re} \subseteq Y\}, \bar{\Re}(Y) = \bigcap\{[y]_{\Re} : [y]_{\Re} \cap Y \neq \varnothing\}$ respectively. Lower and upper approximations are the main openers to the theory of rough set and are used to define other regions using the subset $Y \subseteq U$. For instance, the affirmative, passive, and edge regions of $Y \subseteq U$ are known by lower and upper approximations respectively:

$$POS_{\Re}(Y) = \underline{\Re}(Y)$$
$$NEG_{\Re}(Y) = U - \bar{\Re}(Y)$$
$$BN_{\Re}(Y) = \bar{\Re}(Y) - \underline{\Re}(Y)$$

The set Y is called an exact set relative to $\Re$ if the edge region of it is empty: $BN_{\Re}(Y) = \phi$. Otherwise, the set Y is rough relative to $\Re$. To compare the quality of the results when changing the relation used in the approximation, a tool must be defined that was used with this theory, and this tool is called the accuracy measure. The accuracy measure of the subset $Y \subseteq U$ is known by $\alpha_{\Re}Y = \frac{|\underline{\Re}(Y)|}{|\bar{\Re}(Y)|}$ where $|Y|$ denotes the number of the elements of Y. Clearly $0 \leq \alpha_{\Re}(Y) \leq 1$. If $\alpha_{\Re}(Y) = 1$, then Y is called exact, otherwise, when $\alpha_{\Re}(Y) < 1$, the set is called rough. A topological space is defined as a couple (U, τ) where U is a non-empty set, equipped with a collection τ of subsets of U satisfying the following: $\varnothing, U \in \tau$ and τ is closed when taking an arbitrary union or finite intersection. The topological closure is mentioned by $cl_{\tau}(Y) = \bigcap\{F : Y \subseteq F, F^c \in \tau\}$ for $Y \subset U$ the interior $Y \subset U$ is defined by $int_{\tau}(Y) = \bigcup\{G : G \subseteq Y, G \in \tau\}$.

In this study, we introduce two classes by a general binary relation (not equivalence) R on U. The class $S_{\Re}$ =

$\{\Re_y: y \in U\}$ and $\Re_y = \{x \in U: y\Re x\}$ that generates the topology τ_1 and the class $S_{\Re^\smile} = \{\Re_{\smile y}: y \in U\}$ and $\Re_{\smile y} = \{x \in U: x\Re y\}$ that generates the topology τ_1.

Rough approximations of $Y \subseteq U$ using $S_\Re$ defined in the following: $\underline{\Re}(Y)=\bigcup\{H \in S_\Re : H \subseteq Y\}$ and $\bar{\Re}(Y)=\bigcap\{H \in S_\Re : H \cap Y \neq \emptyset\}$ Also, if $\tau = \tau_1 \cap \tau_2$ we can define two other approximations in the following:

$$\underline{\Re}_\tau(Y)=\bigcup\{H \in \tau : H \subseteq Y\} \text{ and } \bar{\Re}_\tau(Y)=\bigcap\{H \in \tau : H \cap Y \neq \phi\}$$

Control Relation in Interval Information Systems

Interval systems of information (Qian *et al*., 2008) are recognized as a significant kind of information table and it is a reproduction of information systems of one-value. Nowadays, some issues of decision-making in the situation of interval systems of information have been investigated (Bryson and Mobolurin, 1997; Xu and Da, 2002). Mostly, they count on the notion of a potential class amidst two interval numbers (Qian *et al*., 2008; Xu and Da, 2002).

This section is reserved to define a control related to an interval information system (Qian *et al*., 2008) and recall its most significant properties of it.

Table 1: An interval system of information

U	a_1	a_2	a_3	a_4	a_5
Y_1	1	[0,1]	2	1	[1,2]
Y_2	[0,1]	0	[1,2]	0	1
Y_3	[0,1]	0	[1,2]	1	1
Y_4	0	0	1	0	1
Y_5	2	[1,2]	3	[1,2]	[2,3]
Y_6	[0,2]	[1,2]	[1,3]	[1,2]	[2,3]
Y_7	1	1	2	1	2
Y_8	[1, 2]	[1,2]	[2,3]	2	[2,3]
Y_9	[1, 2]	2	[2,3]	[0,2]	3
Y_{10}	2	2	3	[0,1]	3

Definition 4-1

An Interval System of Information (ISI) (Qian *et al*., 2008) is a quadrant $S = (U, CT, W, f)$, such that U consists of non-empty objects that are finite and CT consists of non-empty characteristics that are finite, $W = U_{\alpha \in CT}\ W_\alpha$ and W_α is a domain of characteristic α, $f: U \times CT \rightarrow W$ is a total function such that $f(y, \alpha) \in W_\alpha$ for every $\alpha \in CT$, $y \in U$, termed an information function, where W_α is a set of interval numbers. Termed by:

$$f(y,\alpha) = \left[\alpha^L(y), \alpha^U(y)\right] = \left\{p \mid \alpha^L(y) \le p \le \alpha^U(y), \alpha^L(y), \alpha^U(y) \in R\right\}$$

and termed the interval number of y under the character α. Specifically, $f(y, \alpha)$ would be degraded to a real number if $\alpha^L(y) = \alpha^U(y)$.

With this respect, we deem an information system of one-valued as a particular status of interval systems of information.

Example 4-2

Qian *et al*. (2008) An interval system of information is sitting in Table 1, such that $U = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$, $CT = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

In the dissection of efficient decision-making, it is been always regarded that a double control relation between objects that may be commanding on the one hand of values of a characteristic set in an interval system of information. Generally, a raising priority and a redacting priority are regarded by decision-making. If the domain of a character is ordered counting on a raising or redacting priority, then the character is a standard.

In case all attributes are criteria, then an interval system of information is said to be an Interval Ordered System of Information (IOSI).

It is supposed that the domain of standard $\alpha \in CT$ is predemanded by the relation $\preceq_\alpha$: $y \preceq_\alpha x \Leftrightarrow f(y, \alpha) \le f(x, \alpha)$ (depending on a raising priority) or $\succeq_\alpha$: $y \succeq_\alpha x \Leftrightarrow f(y, \alpha) \ge f(x, \alpha)$ (depending on a reduced priority), where $y, x \in U$. Thus $y \succeq_\alpha x$ means that "y is at least as good as x relative to criterion α". We say that y controls x relative to a subset of characteristics $A \subseteq CT$, (or, y A-controls x), termed by $y \succeq_A x$, if $y \succeq_\alpha x$, $\forall\ \alpha \in A$. That is, "y is at least as good as x relative to all criteria in A". Since the intersection of complete pre-orders is a partial pre-order, the control relation $\succeq_A$ is a partial pre-order.

Now we recall a control relation (Qian *et al*., 2008) that distinguishes control degrees from an interval-ordered system of information. In an IOSI, we say that y controls x relative to $A \subseteq CT$ *if* y $\succeq_A x$ and termed by $y\,\Re_A^{\ge} x$. That is $\Re_\ge = \{(x, y) \in U \times U: x \succeq_\alpha y\}$ Similarly, recall a controlled relation $\Re^\le$ in the following: $\Re_\le = \{(x, y) \in U \times U: y \succeq_\alpha x\}$.

Given $A \subseteq CT$ and $A = A_1 \cup A_2$, where the characteristics set A_1 depending on raising priority and A_1 depending on redacting priority. Now redefine these two double relations more accurately in the following:

$$\begin{aligned}\Re_A^{\ge} = &\left\{(x,y) \in U \times U : \alpha_1^L(x) \ge \alpha_1^L(y), \alpha_1^U(x) \ge \alpha_1^U(y)\ (\forall \alpha_1 \in A_1); \alpha_2^L(x) \ge \alpha_2^L(y), \alpha_2^U(x) \ge \alpha_2^U(y)\ (\forall \alpha_2 \in A_2)\right\} \\ = &\left\{(x,y) \in U \times U : (x,y) \in \Re_A^{\ge}\right\}\end{aligned}$$

$$\begin{aligned}\Re_A^{\le} = &\left\{(x,y) \in U \times U : \alpha_1^L(x) \le \alpha_1^L(y), \alpha_1^U(x) \le \alpha_1^U(y)\ (\forall \alpha_1 \in A_1); \alpha_2^L(x) \ge \alpha_2^L(y), \alpha_2^U(x) \ge \alpha_2^U(y)\ (\forall \alpha_2 \in A_2)\right\} \\ = &\left\{(x,y) \in U \times U : (x,y) \in \Re_A^{\le}\right\}\end{aligned}$$

Based on the notion of $\Re_A^{\geq}$ and $\Re_A^{\leq}$, the Properties presented here are not hard to obtain.

Proposition 4-3

Qian *et al*. (2008) suppose that $S = (U, CT, W, f)$ is an IOSI, $Y \subseteq U, A \subseteq CT$, and $\Re_A^{\geq}$ a control relation, then:

$$\Re_A^{\geq} = \bigcap_{a \in A} \Re_{\{a\}}^{\geq},\ \Re_A^{\leq} = \bigcap_{a \in A} \Re_{\{a\}}^{\leq}$$

Proposition 4-4

Qian *et al*. (2008) suppose that $S = (U, CT, W, f)$ is an IOSI, $Y \subseteq U, A \subseteq CT$, and $\Re_A^{\geq}$ a control relation, then:

- $\Re_A^{\geq}$, $\Re_A^{\leq}$ are reflexive
- $\Re_A^{\geq}$, $\Re_A^{\leq}$ are antisymmetric and
- $\Re_A^{\geq}$, $\Re_A^{\leq}$ are transitive

The control class induced by the control relation $\Re^{\geq}{}_A$ is the set of objects controlling x, i.e.:

$$\begin{aligned}[y]_A^{\geq} = \{x \in U : &\alpha_1^L(x) \geq \alpha_1^L(y), \alpha_1^U(x) \\ &\geq \alpha_1^U(y)(\forall \alpha_1 \in A_1); \alpha_2^L(x) \\ &\geq \alpha_2^L(y), \alpha_2^U(x) \geq \alpha_2^U(y)(\forall \alpha_2 \in A_2)\} \\ &= \{x \in U : (x, y) \in \Re_A^{\geq}\}\end{aligned}$$

and the set of objects controlled by y:

$$\begin{aligned}[y]_A^{\leq} = \{x \in U : &\alpha_1^L(x) \leq \alpha_1^L(y), \alpha_1^U(x) \\ &\leq \alpha_1^U(y)(\forall \alpha_1 \in A_1); \alpha_2^L(x) \\ &\leq \alpha_2^L(y), \alpha_2^U(x) \geq \alpha_2^U(y)(\forall \alpha_2 \in A_2)\} \\ &= \{x \in U : (x, y) \in \Re_A^{\leq}\}\end{aligned}$$

where, $[y]_A^{\geq}$ expressed the set of objects that may control y and $[y]_A^{\leq}$ describes the set of objects that may be controlled by y in terms of A in an interval-ordered system of information, which are termed the A-controlling set and the A-controlling set relative to $y \in U$, respectively.

Simply, without any loss of generality, only characteristics with redacting priority are regarded here. The proof of the following Proposition is found in (Qian *et al*., 2008).

Proposition 4-5

Suppose that $S = (U, CT, W, f)$ *is an IOSI*, $Y \subseteq U$, A, $B \subseteq CT$, then:

- If $B \subseteq A \subseteq CT$, then $\Re_B^{\geq} \subseteq \Re_A^{\geq} \subseteq \Re_{CT}^{\geq}$
- If $B \subseteq A \subseteq CT$, then $[y]_B^{\geq} \subseteq [y]_A^{\geq} \subseteq [y]_{CT}^{\geq}$
- If $y_j \in [y_i]_A^{\geq}$ than $[y_j]_A^{\geq} \subseteq [y_i]_A^{\geq}$ and $[y_i]_A^{\geq} = \bigcup \{[y_j]_A^{\geq} : y_j \in [y_i]_A^{\geq}\}$ and
- $[y_i]_A^{\geq} = [y_i]_A^{\geq}$ if and only if $f(y_i, \alpha) = f(y_j, \alpha)$ $(\forall \alpha \in A)$

Let $U / \Re_{CT}^{\geq}$ refer to the classification of the universe, which is defined to be $\{[y]_A^{\geq} | y \in U\}$. Any element from $U / \Re_A^{\geq}$ will be termed a control class relative to A.

Control classes $U / \Re_A^{\geq}$ do not comprise a subdivision of U in general. They comprise a covering of U. These concepts and properties presented above can be clarified through the following example.

Example 4-6

Counting the ranking induced by the control relation $\Re_A^{\geq}$ in Table 1 considering Example 4.2. Based on Table 1, the following is obtained:

$$U / \Re_{CT}^{\geq} = \{[y_1]_{CT}^{\geq}, [y_2]_{CT}^{\geq}, [y_3]_{CT}^{\geq}, \ldots, [y_{10}]_{CT}^{\geq}\}$$

such that:

$$\begin{aligned}
&[y_1]_{CT}^{\geq} = \{y_1, y_5, y_7, y_8\} \\
&[y_2]_{CT}^{\geq} = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\} \\
&[y_3]_{CT}^{\geq} = \{y_1, y_3, y_5, y_6, y_7, y_8\} \\
&[y_4]_{CT}^{\geq} = \{y_1, y_2, y_3\, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\} \\
&[y_5]_{CT}^{\geq} = \{y_5\}, [y_6]_{CT}^{\geq} = \{y_5, y_6, y_8\} \\
&[y_7]_{CT}^{\geq} = \{y_5, y_7, y_8\}, [y_8]_{CT}^{\geq} = \{y_8\} \\
&[Y_9]_{CT}^{\geq} = \{y_9\}, [y_{10}]_{CT}^{\geq} = \{y_{10}\}
\end{aligned}$$

and:

$$U / \Re_{CT}^{\leq} = \{[y_1]_{CT}^{\leq}, [y_2]_{CT}^{\leq}, [y_3]_{CT}^{\leq}, \ldots, [y_{10}]_{CT}^{\leq}\}$$

where:

$$\begin{aligned}
&[y_1]_{CT}^{\leq} = \{y_1, y_2, y_3, y_4\}, [y_2]_{CT}^{\leq} = \{y_2, y_4\} \\
&[y_3]_{CT}^{\leq} = \{y_2, y_3, y_4\}, [y_4]_{CT}^{\geq} = \{y_4\} \\
&[y_5]_{CT}^{\leq} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\} \\
&[y_6]_{CT}^{\leq} = \{y_2, y_3, y_4, y_6\} \\
&[y_7]_{CT}^{\geq} = \{y_1, y_2, y_3, y_4, y_7\}, \\
&[y_8]_{CT}^{\geq} = \{y_1, y_2, y_3, y_4, y_6, y_7, y_8\} \\
&[y_9]_{CT}^{\geq} = \{y_2, y_4, y_9\}, [y_{10}]_{CT}^{\geq} = [y_2, y_4, y_{10}]
\end{aligned}$$

Regarding the above example, it is not difficult to verify (3) of Proposition 4.5. Control degrees $U/\Re_{CT}^{\geq}$ comprise a covering of *U***.** In specific implementations, various control relations depending on different indications analysis are defined, and the corresponding control class of each object relative to some control relation is obtained.

Generalized Covering Approximation Space and its Applications to Interval Ordered Information Systems

In this section, we investigate the case of covering approximation (Abd El-Monsef *et al*., 2015) relative to a control relation $\Re_A^{\geq}$ in interval-ordered systems of information, and to do that we need to introduce a notion of right (left) covering approximation space relative to a control relation $\Re_A^{\geq}$ ($\Re_A^{\leq}$ respectively).

The following definition is induced from (Abd El-Monsef *et al*., 2015) and redefined here relative to a control relation.

Definition 5-1

Suppose that $S = (U, CT, W, f)$ is an IOSI and $A \subseteq CT$. Then, the after set (resp. the fore set) of the element $y \in U$ relative to the control relation $\Re_A^{\geq}$ (resp. $\Re_A^{\leq}$) is the class:

$$y\Re_A^{\geq} = \left\{x \in U : x \succeq_A y\right\}$$
$$\left(resp. \Re_A^{\leq} y = \left\{x \in U : x \preceq_A y\right\}\right)$$

Example 5-2

Compute the after set and the fore set by the control relation $\Re_A^{\geq}$ in Table 1 considering Example 4.2. From Table 1, one can get that
and:

$$y_1\Re_{CT}^{\geq} = \left\{y_1, y_5, y_7, y_8\right\}$$
$$y_2\Re_{CT}^{\geq} = \left\{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\right\}$$
$$y_3\Re_{CT}^{\geq} = \left\{y_1, y_3, y_5, y_6, y_7, y_8\right\}$$
$$y_4\Re_{CT}^{\geq} = \left\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\right\}$$
$$y_5\Re_{CT}^{\geq} = \left\{y_5\right\}, y_6\Re_{CT}^{\geq} = \left\{y_5, y_6, y_8\right\}$$
$$y_7\Re_{CT}^{\geq} = \left\{y_5, y_7, y_8\right\}, y_8\Re_{CT}^{\geq} = \left\{y_8\right\}$$
$$y_9\Re_{CT}^{\geq} = \left\{y_9\right\}, y_{10}\Re_{CT}^{\geq} = \left\{y_{10}\right\}$$

and:

$$y_1\Re_{CT}^{\leq} = \left\{y_1, y_2, y_3, y_4\right\}, y_2\Re_{CT}^{\leq} = \left\{y_2, y_4\right\}$$
$$y_3\Re_{CT}^{\leq} = \left\{y_2, y_3, y_4\right\}, y_4\Re_{CT}^{\leq} = \left\{y_4\right\}$$
$$y_5\Re_{CT}^{\leq} = \left\{y_1, y_2, y_3, y_4, y_5, y_6, y_7\right\}$$
$$y_6\Re_{CT}^{\leq} = \left\{y_2, y_3, y_4, y_6\right\}$$
$$y_7\Re_{CT}^{\leq} = \left\{y_1, y_2, y_3, y_4, y_7\right\},$$
$$y_8\Re_{CT}^{\leq} = \left\{y_1, y_2, y_3, y_4, y_6, y_7, y_8\right\}$$
$$y_9\Re_{CT}^{\leq} = \left\{y_2, y_4, y_9\right\}, y_{10}\Re_{CT}^{\leq} = \left\{y_2, y_4, y_{10}\right\}$$

Here we define two types of covering induced by the concept of after and fore sets.

Definition 5-3

Suppose that $S = (U, CT, W, f)$ is an IOSI and $A \subseteq CT$. The two types of coverage for U induced by the concept of after sets and fore set relative to the control relation $\Re_A^{\geq}$ (resp. $\Re_A^{\geq}$) are defined here:

1. Right covering ($r^{\geq}$-covering) $C^{\geq} = \left\{y\Re_A^{\geq} : y \in U\right\}$ such that

$$U = \bigcup_{y\in U} y\Re_A^{\geq}$$

2. Left covering ($l^{\leq}$-covering) $C^{\leq} = \left\{\Re_A^{\leq} y : y \in U\right\}$ such that

$$U = \bigcup_{y\in U} \Re_A^{\leq} y$$

Example 5-4

The right covering ($r^{\geq}$-covering) and the left covering ($l^{\leq}$-covering) by the control relation $\Re_{CT}^{\geq}$ (resp. $\Re_{CT}^{\leq}$) in Table 1 considering Example 4.2 are calculated:

$$C_{CT}^{\geq} = \left\{\left\{y_1, y_5, y_7, y_8\right\}, \left\{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\right\}, \left\{y_1, y_3, y_5, y_6, y_7, y_8\right\}, \left\{y_5\right\}, \left\{y_5, y_6, y_8\right\}, \left\{y_5, y_7, y_8\right\} \left\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\right\}, \left\{y_8\right\}, \left\{y_9\right\}, \left\{y_{10}\right\}\right\}$$

and:

$$C_{CT}^{\leq} = \left\{\left\{y_1, y_2, y_3, y_4\right\}, \left\{y_2, y_4\right\}, \left\{y_2, y_3, y_4\right\}, \left\{y_4\right\}, \left\{y_2, y_3, y_4,, y_6\right\}, \left\{y_1, y_2, y_3, y_4, y_5, y_6, y_7,\right\}, \left\{y_1, y_2, y_3, y_4, y_7\right\}, \left\{y_1, y_2, y_3, y_4, y_6, y_7, y_8\right\}, \left\{y_2, y_4, y_9\right\}, \left\{y_2, y_4, y_{10}\right\}\right\}$$

It is clear that:

$$U = \bigcup_{y\in U} y\Re_{CT}^{\geq} \text{ and } U = \bigcup_{y\in U} \Re_{CT}^{\leq} y$$

Lemma 5-5

Suppose that $S = (U, CT, W, f)$ is an IOSI and $A \subseteq$ CT and the control relations $\mathfrak{R}_A^{\geq}$ (resp. $\mathfrak{R}_A^{\leq}$) be serial relations on U. Then we get:

$$U = \bigcup_{y \in U} y\mathfrak{R}_A^{\geq} \text{ and } U = \bigcup_{y \in U} \mathfrak{R}_A^{\leq} y$$

From the above Lemma, we can notice that if $\mathfrak{R}_{CT}^{\geq}$ (resp. $\mathfrak{R}_{CT}^{\leq}$) be serial relations on U, then $y\mathfrak{R}_{CT}^{\geq}$ (resp. $\mathfrak{R}_{CT}^{\leq}y$) exemplifies a right covering (resp. left covering) of U relative to the control relation $\mathfrak{R}_{CT}^{\geq}$ (resp. $\mathfrak{R}_{CT}^{\leq}$).

Definition 5-6

Suppose that $S = (U, CT, W, f)$ is an IOSI, $A \subseteq CT$, the control relations $\mathfrak{R}_A^{\geq}$ (resp. $\mathfrak{R}_A^{\geq}$) defined on U and $\mathfrak{C}$ be an $r^{\geq}$-covering (resp. an $l^{\leq}$-covering) of U associated to $\mathfrak{R}_A^{\geq}$ (resp. $\mathfrak{R}_A^{\geq}$). Then the triple $\langle S, \mathfrak{R}_A^{\geq}, \mathfrak{C} \rangle$ (resp. $\langle S, \mathfrak{R}_A^{\leq}, \mathfrak{C} \rangle$) is called $\mathfrak{H}_n^{\geq}$ -right covering approximation space relative to the control relation $\mathfrak{R}_A^{\geq}$ (resp. $\mathfrak{H}_n^{\leq}$-left covering approximation space relative to the control relation $\mathfrak{R}_A^{\leq}$) (briefly, , $\mathfrak{H}_n^{\geq}$ -RCAS (resp. $\mathfrak{H}_n^{\leq}$ -LCAS)).

Definition 5-7

Let $\langle S, \mathfrak{R}_A^{\geq}, \mathfrak{C} \rangle$ (resp. $\langle S, \mathfrak{R}_A^{\leq}, \mathfrak{C} \rangle$) be a $\mathfrak{H}_n^{\geq}$ -RCAS (resp. $\mathfrak{H}_n^{\leq}$ -LCAS). Then for every element $y \in U$, we define four types of neighborhoods here:

$$r^{\geq} - nieghborhood : N_{r^{\geq}}(y) = \bigcap \{ K \in C^{\geq} : y \in K \}$$
$$l^{\leq} - nieghborhood : N_{l^{\leq}}(y) = \bigcap \{ K \in C^{\leq} : y \in K \}$$
$$i^{\leq} - nieghborhood : N_{i^{\leq}}(y) = N_{r^{\geq}}(y) \bigcap N_{l^{\leq}}(y)$$
$$u^{\geq} - nieghborhood : N_{u^{\leq}}(y) = N_{r^{\geq}}(y) \bigcup N_{l^{\leq}}(y)$$

Example 5-8

The four types of neighborhoods by the control relation $\mathfrak{R}_A^{\geq}$ (resp. $\mathfrak{R}_A^{\geq}$) in Table 1 considering Example 4.2 can be calculated as presented here.

The $r^{\geq}$-neighborhoods:

$$N_{(r^{\geq},CT)}(y_1) = \{y_1, y_5, y_7, y_8\}$$
$$N_{(r^{\geq},CT)}(y_2) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$$
$$N_{(r^{\geq},CT)}(y_3) = \{y_1, y_3, y_5, y_6, y_7, y_8\}$$
$$N_{(r^{\geq},CT)}(y_4) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$$
$$N_{(r^{\geq},CT)}(y_5) = \{y_5\}, N_{(r^{\geq},CT)}(y_6) = \{y_5, y_6, y_8\}$$
$$N_{(r^{\geq},CT)}(y_7) = \{y_5, y_7, y_8\}, N_{(r^{\geq},CT)}(y_8) = \{y_8\}$$
$$N_{(r^{\geq},CT)}(y_9) = \{y_9\}, N_{(r^{\geq},CT)}(y_{10}) = \{y_{10}\}$$

the $l^{\leq}$- neighborhoods:

$$N_{(l^{\leq},CT)}(y_1) = \{y_1, y_2, y_3, y_4\}, N_{(l^{\leq},CT)}(y_2) = \{y_2, y_4\}$$
$$N_{(l^{\leq},CT)}(y_3) = \{y_2, y_3, y_4\}, N_{(l^{\leq},CT)}(y_4) = \{y_4\}$$
$$N_{(l^{\leq},CT)}(y_5) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$$
$$N_{(l^{\leq},CT)}(y_6) = \{y_2, y_3, y_4, y_6\},$$
$$N_{(l^{\leq},CT)}(y_7) = \{y_1, y_2, y_3, y_4, y_7\},$$
$$N_{(l^{\leq},CT)}(y_8) = \{y_1, y_2, y_3, y_4, y_6, y_7, y_8\}$$
$$N_{(l^{\leq},CT)}(y_9) = \{y_2, y_4, y_9\}, N_{(l^{\leq},CT)}(y_{10}) = \{y_2, y_4 y_{10}\}$$

the $i^{\leq}$ -neighborhoods:

$$N_{(i^{\leq},CT)}(y_1) = \{y_1\}, N_{(i^{\leq},CT)}(y_2) = \{y_2\}$$
$$N_{(i^{\leq},CT)}(y_3) = \{y_3\}, N_{(i^{\leq},CT)}(y_4) = \{y_4\}$$
$$N_{(i^{\leq},CT)}(y_5) = \{y_5\}, N_{(i^{\leq},CT)}(y_6) = \{y_6\}$$
$$N_{(i^{\leq},CT)}(y_7) = \{y_7\}, N_{(i^{\leq},CT)}(y_8) = \{y_8\}$$
$$N_{(i^{\leq},CT)}(y_9) = \{y_9\}, N_{(i^{\leq},CT)}(y_{10}) = \{y_{10}\}$$

and the $u^{\geq}$ -neighborhoods:

$$N_{(u^{\geq},CT)}(y_1) = \{y_1, y_2, y_3, y_4, y_5, y_7, y_8\}$$
$$N_{(u^{\geq},CT)}(y_2) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$$
$$N_{(u^{\geq},CT)}(y_3) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$$
$$N_{(u^{\geq},CT)}(y_4) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$$
$$N_{(u^{\geq},CT)}(y_5) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$$
$$N_{(u^{\geq},CT)}(y_6) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$$
$$N_{(u^{\geq},CT)}(y_7) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_8\}$$
$$N_{(u^{\geq},CT)}(y_8) = \{y_1, y_2, y_3, y_4, y_6, y_7, y_8\}$$
$$N_{(u^{\geq},CT)}(y_9) = \{y_2, y_4, y_9\}$$
$$N_{(u^{\geq},CT)}(y_{10}) = \{y_2, y_4, y_{10}\}$$

Definition 5-9

Let $\langle S, \mathfrak{R}_A^{\geq}, \mathfrak{C} \rangle$ (resp. $\langle S, \mathfrak{R}_A^{\leq}, \mathfrak{C} \rangle$) be a $\mathfrak{H}_n^{\geq}$ - -RCAS (resp. $\mathfrak{H}_n^{\leq}$ -LCAS). and $Y \subseteq U$. Then the j-lower approximation and j-upper approximation of Y relative to the control relations $\Re_A^{\geq}$ and $\Re_A^{\leq}$ are defined respectively:

$$\left(\underline{\Re}_j\right)_A^{\geq}(Y)=\left\{y\in Y: N_{j\geq}(y)\subseteq Y\right\}$$
$$\left(\overline{\Re}_j\right)_A^{\geq}(Y)=\left\{y\in Y: N_{j\geq}(y)\cap Y\neq\phi\right\}$$

where: $j \in \{r, u\}$:

$$(\underline{\Re}_j)_A^{\leq}(Y)=\{y\in Y: N_{j\leq}(y)\subseteq Y\}$$
$$(\overline{\Re}_j)_A^{\leq}(Y)=\{y\in Y: N_{j\leq}(y)\cap Y\neq\emptyset\}$$

Where, $j \in \{i, l\}$.

We observe, from Definition 5.9, that $\left(\underline{\Re}_j\right)_A^{\geq}(Y)$ and $\left(\overline{\Re}_j\right)_A^{\geq}(Y),\left(j\in\{r, u\}\right)$ are the sets of all objects that are certainly included in Y and $\left(\underline{\Re}_j\right)_A^{\leq}(Y)$ and $\left(\overline{\Re}_j\right)_A^{\leq}(Y)\left(j\in\{i, l\}\right)$ are the set of objects that possibly contained in Y. $\left(B_j\right)_A^{\geq}(Y)=\left(\overline{\Re}_j\right)_A^{\geq}(Y)-\left(\underline{\Re}_j\right)_A^{\geq}(Y)$, refers to the right j-Boundary of the approximation relative to the control relation $\Re_A^{\geq}$ where $j \in \{r, u\}$ and $\left(B_j\right)_A^{\leq}(Y)=\left(\underline{\Re}_j\right)_A^{\leq}(Y)-\left(\overline{\Re}_j\right)_A^{\leq}(Y)$, refers to the left j-Boundary of the approximation relative to the control relation $\Re_A^{\leq}$ where $j \in \{l, i\}$.

The following properties are easy can be checked.

Proposition 5-10

Suppose that $S = (U, CT, W, f)$ is an IOSI, $Y \subseteq U$, $A \subseteq CT$, and $\Re_A^{\geq}$ a control relation, then:

$$1-\left(\underline{\Re}_j\right)_A^{\geq}(\phi)=(\overline{\Re}_j)_A^{\geq}(\phi)=\phi$$
$$2-\left(\underline{\Re}_j\right)_A^{\geq}(U)=(\overline{\Re}_j)_A^{\geq}(U)=U$$
$$3-(\underline{\Re}_j)_A^{\geq}(Y)\subseteq Y\subseteq\left(\overline{\Re}_j\right)_A^{\geq}(Y)$$
$$4-(\underline{\Re}_j)_A^{\geq}\left((\underline{\Re}_j)_A^{\geq}(Y)\right)=(\underline{\Re}_j)_A^{\geq}(Y)\left(\overline{\Re}_j\right)_A^{\geq}\left(\left(\overline{\Re}_j\right)_A^{\geq}\right)(Y)=\left(\overline{\Re}_j\right)_A^{\geq}(Y)$$
$$5-(\underline{\Re}_j)_A^{\geq}(Y)\subseteq\left(\underline{\Re}_j\right)_{CT}^{\geq}(Y), and \left(\overline{\Re}_j\right)_A^{\geq}(Y)\subseteq(\overline{\Re}_j)_{CT}^{\geq}(Y)$$
$$6-(\underline{\Re}_j)_A^{\geq}(Y)=\sim(\overline{\Re}_j)_A^{\geq}(A), and \left(\overline{\Re}_j\right)_A^{\geq}(Y)\sim\left(\underline{\Re}_j\right)_A^{\geq}(Y)$$

where, $j \in \{r, u\}$.

Note that Proposition 5.10 is correct for a control relation $(\Re_j)_A^{\leq}$ where $j \in \{l, i\}$.

Proposition 5-11

Suppose that $S = (U, CT, W, f)$ is an IOSI, $Y \subseteq U$, $A \subseteq CT$, and $\Re_A^{\geq}$ a control relation, then:

$$-if\ Y\subseteq X\ then: \left(\underline{\Re}_j\right)_A^{\geq}(Y)\subseteq(\underline{\Re}_j)_A^{\geq}(X),(\overline{\Re}_j)_A^{\geq}(Y)\subseteq(\overline{\Re}_j)_A^{\geq}(X)$$
$$-\left(\underline{\Re}_j\right)_A^{\geq}(X\cap Y)=(\overline{\Re}_j)_A^{\geq}(X)\cap(\overline{\Re}_j)_A^{\geq}(Y),$$
$$(\underline{\Re}_j)_A^{\geq}(X\cup Y)=(\overline{\Re}_j)_A^{\geq}(X)\cup\left(\overline{\Re}_j\right)_A^{\geq}(Y)$$
$$-(\overline{\Re}_j)_A^{\geq}(X\cap Y)=(\underline{\Re}_j)_A^{\geq}(X)\cap\left(\overline{\Re}_j\right)_A^{\geq}(Y),$$
$$(\overline{\Re}_j)_A^{\geq}(X\cup Y)=\left(\underline{\Re}_j\right)_A^{\geq}(X)\cup\left(\overline{\Re}_j\right)_A^{\geq}(Y)$$

where, $j \in \{r, u\}$. $\left(\underline{\Re}_j\right)_A^{\geq}$

Note that Proposition 5.11 is correct for a control relation $\left(\Re_j\right)_A^{\leq}$ where $j \in \{l, i\}$.

The j-lower and j-upper approximations of Y relative to the control relation $\left(\Re_j\right)_A^{\geq}$ where $j \in \{r, u\}$. (resp. $\left(\Re_j\right)_A^{\leq}$ where $j \in \{l, i\}$) can be used to recap control bases by a decision-maker, where control rules are reproduced with sureness by using $\left(\Re_j\right)_A^{\geq}(Y)$, $j \in \{r, u\}$ (resp. $\left(\Re_j\right)_A^{\leq}(Y)$, $j \in \{l, i\}$) and can reproduce possible control rules by using $\left(B_j\right)_A^{\geq}(Y)=\left(\overline{\Re}_j\right)_A^{\geq}(Y)$-$\left(\underline{\Re}_j\right)_A^{\geq}(Y)$, where $j \in \{r, u\}$ (resp. $\left(B_j\right)_A^{\leq}(Y)$=$\left(\overline{\Re}_j\right)_A^{\leq}(Y)$-$\left(\underline{\Re}_j\right)_A^{\leq}(Y)$ where $j \in \{l, i\}$).

Proposition 5-12

Suppose that $S = (U, CT, W, f)$ is an IOSI, $A \subseteq CT$. If $\left(\Re_j\right)_A^{\geq}=\left(\Re_j\right)_{CT}^{\geq}$, then $\left(\underline{\Re}_j\right)_A^{\geq}(Y)=\left(\underline{\Re}_j\right)_{CT}^{\geq}(Y)$ and $\left(\overline{\Re}_j\right)_A^{\geq}(Y)=\left(\overline{\Re}_j\right)_{CT}^{\geq}(Y)$, where $j \in \{r, u\}$.

Note that proposition 5.12 is correct for a control relation $\left(\Re_j\right)_A^{\leq}$ where $j \in \{l, i\}$.

Uncertainty of the covering approximation is dependent on the subsistence of a boundary region. So, the higher the boundary region of a covering approximation, the lower the accuracy of the covering approximation. To calculate the imprecision of a covering approximation induced by control relation $\left(\Re_j\right)_A^{\geq}(Y)$ in an interval-ordered system of information, we present a notion of accuracy measure in the following.

Definition 5-13

Suppose that $S = (U, CT, W, f)$ is an IOSI, $Y \subseteq U$, $A \subseteq CT$.

A right j-accuracy measure of Y relative to the control relation $\left(\Re_j\right)_A^{\geq}$ is defined as follows here:

$$\alpha\left((\mathfrak{R}_j)_A^{\geq}, Y\right) = \frac{\left|\left(\underline{\mathfrak{R}}_j\right)_A^{\geq}(Y)\right|}{\left|\left(\overline{\mathfrak{R}}_j\right)_A^{\geq}(Y)\right|} = \frac{\left|\left(\underline{\mathfrak{R}}_j\right)_A^{\geq}(Y)\right|}{|U| - \left|\left(\underline{\mathfrak{R}}_j\right)_A^{\geq}(\sim Y)\right|}$$

where, $j \in \{r, u\}$.

Note that one can define a left j-accuracy measure of Y relative to the control relation $(\mathfrak{R}_j)_A^{\leq}$ where $j \in \{l, i\}$ similar to Definition 5.13.

The right j-accuracy measure explains the class of completeness of the knowledge about Y, given the subdivisions of $U/(\underline{\mathfrak{R}}_j)_A^{\geq}$. It is trivial to notice that this measure not only counts on the j-lower approximation of Y but also on the j-lower approximation of $\sim Y$.

The following result is induced from Definition 5.13 and Proposition 5.12.

Corollary 5-14

Suppose that $S = (U, CT, W, f)$ is an IOSI, $A \subseteq CT$. If $(\mathfrak{R}_j)_A^{\geq} = (\mathfrak{R}_j)_{CT}^{\geq}$, then $\alpha\left((\mathfrak{R}_j)_A^{\geq}(Y)\right) = \alpha\left((\mathfrak{R}_j)_{CT}^{\geq}(Y)\right)$.

Proposition 5-15

Suppose that $S = (U, CT, W, f)$ is an IOSI, $Y \subseteq U$, and $B \subseteq A \subseteq CT$, then:

$$\alpha\left((\mathfrak{R}_j)_{CT}^{\geq}(Y)\right) \geq \alpha\left((\mathfrak{R}_j)_A^{\geq}\right) \geq \alpha\left((\mathfrak{R}_j)_B^{\geq}(Y)\right)$$

Proof since $A \subseteq CT$, the by (5) of Proposition 5.10 we have $(\underline{\mathfrak{R}}_j)_A^{\geq}(Y) \subseteq (\underline{\mathfrak{R}}_j)_{CT}^{\geq}(Y), (\overline{\mathfrak{R}}_j)_A^{\geq}(Y) \subseteq (\overline{\mathfrak{R}}_j)_{CT}^{\geq}(Y)$. Therefore:

$$\alpha\left((\mathfrak{R}_j)_A^{\geq}, Y\right) = \frac{\left|\left(\underline{\mathfrak{R}}_j\right)_A^{\geq}(Y)\right|}{\left|\left(\overline{\mathfrak{R}}_j\right)_A^{\geq}(Y)\right|} \leq \frac{\left|\left(\underline{\mathfrak{R}}_j\right)_{CT}^{\geq}(Y)\right|}{\left|\left(\overline{\mathfrak{R}}_j\right)_A^{\geq}(Y)\right|} = \alpha\left((\mathfrak{R}_j)_{CT}^{\geq}, Y\right)$$

that is, $\alpha\left((\mathfrak{R}_j)_A^{\geq}, Y\right) \leq \alpha\left((\mathfrak{R}_j)_{CT}^{\geq}, Y\right)$. On the same way, we get $\alpha\left((\mathfrak{R}_j)_B^{\geq}, Y\right) \leq \alpha\left((\mathfrak{R}_j)_A^{\geq}, Y\right)$, and we are done.

Example 5-16

The j-lower approximation and j-upper approximation of Y relative to the control relations $\mathfrak{R}_{CT}^{\geq}$ and $\mathfrak{R}_{CT}^{\leq}$ in Table 1 considering Example 4.2 can be calculated as in the following: Let $Y = \{y_1, y_5, y_6, y_8\}$, then:

$$\underline{\mathfrak{R}}^{(\geq_r, CT)}(Y) = \{y_5, y_6, y_8\}$$
$$\overline{\mathfrak{R}}^{(\geq_r, CT)}(Y) = \{y_1, y_2, y_3\, y_4\, y_5, y_6, y_7, y_8\}$$
$$\underline{\mathfrak{R}}^{(\geq_l, CT)}(Y) = \phi,\ \overline{\mathfrak{R}}^{(\geq_r, CT)}(Y) = \{y_1, y_5, y_6, y_7, y_8\}$$
$$\underline{\mathfrak{R}}^{(\geq_i, CT)}(Y) = \{y_1, y_5, y_6, y_8\}$$
$$\overline{\mathfrak{R}}^{(\geq_i, CT)}(Y) = \{y_1, y_5, y_6, y_8\}$$
$$\underline{\mathfrak{R}}^{(\geq_u, CT)}(Y) = \phi$$
$$\overline{\mathfrak{R}}^{(\geq_u, CT)}(Y) = \{y_1, y_2, y_3\, y_4\, y_5, y_6, y_7, y_8\}$$

The right j-accuracy measure and the left j-accuracy measure of Y relative to the control relation $(\mathfrak{R}_j)_{CT}^{\geq}$ and $(\mathfrak{R}_j)_{CT}^{\leq}$ can be calculated as in the following:

$$\alpha\left((\mathfrak{R}_r)_{CT}^{\geq}, Y\right) = \frac{\left|\left(\underline{\mathfrak{R}}_r\right)_{CT}^{\geq}(Y)\right|}{\left|\left(\overline{\mathfrak{R}}_r\right)_{CT}^{\geq}(Y)\right|} = \frac{3}{8}$$

$$\alpha\left((\mathfrak{R}_l)_{CT}^{\leq}, Y\right) = \frac{\left|\left(\underline{\mathfrak{R}}_l\right)_{CT}^{\leq}(Y)\right|}{\left|\left(\overline{\mathfrak{R}}_l\right)_{CT}^{\leq}(Y)\right|} = 0$$

$$\alpha\left((\mathfrak{R}_i)_{CT}^{\leq}, Y\right) = \frac{\left|\left(\underline{\mathfrak{R}}_i\right)_{CT}^{\leq}(Y)\right|}{\left|\left(\overline{\mathfrak{R}}_i\right)_{CT}^{\leq}(Y)\right|} = 1$$

$$\alpha\left((\mathfrak{R}_u)_{CT}^{\geq}, Y\right) = \frac{\left|\left(\underline{\mathfrak{R}}_u\right)_{CT}^{\geq}(Y)\right|}{\left|\left(\overline{\mathfrak{R}}_u\right)_{CT}^{\geq}(Y)\right|} = 0$$

Definition 5-17

Let $S, \mathfrak{R}_A^{\geq}, \mathfrak{E}$ (resp. $\langle S, \mathfrak{R}_A^{\leq}, \mathfrak{E}\rangle$) be a $\mathfrak{H}_n^{\geq}$ -RCAS (resp. $\mathfrak{H}_n^{\leq}$ -LCAS). Then the following families give the topologies associated with the $\mathfrak{H}_n^{\geq}$ -RCAS (resp. $\mathfrak{H}_n^{\leq}$ -LCAS).

$$\tau_j^{\geq} = \{Y \subseteq U : \forall y \in Y, N_{j\geq}(y) \subseteq Y, \forall j \in \{r, u\}\}$$
$$\tau_j^{\leq} = \{Y \subseteq U : \forall y \in Y, N_{j\geq}(y) \subseteq Y, \forall j \in \{i, l\}\}$$

Example 5-18

The topologies induced from the neighborhoods relative to the control relations $\mathfrak{R}_{CT}^{\geq}$ and $\mathfrak{R}_{CT}^{\leq}$ in Table 1 considering Example 4.2 can be calculated as presented here:

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$$\tau_r^{(\geq,CT)} = \{U, \phi, \{y_5\}, \{y_8\}, \{y_9\}, \{y_{10}\}, \{y_5, y_8\}, \{y_5, y_9\},$$
$$\{y_5, y_{10}\}, \{y_5, y_8, y_9\}, \{y_5, y_6, y_8\}, \{y_5, y_8, y_{10}\}, \{y_5, y_9, y_{10}\},$$
$$\{y_8, y_9, y_{10}\}, \{y_5, y_7, y_8\}, \{y_5, y_8, y_9, y_{10}\}, \{y_5, y_6, y_8, y_9\},$$
$$\{y_5, y_7, y_8, y_9\}, \{y_5, y_6, y_8, y_{10}\}, \{y_5, y_7, y_8, y_{10}\}, \{y_5, y_6, y_7, y_8,\},$$
$$\{y_5, y_6, y_8, y_9, y_{10}\}, \{y_5, y_7, y_8, y_9, y_{10}\}, \{y_5, y_6, y_7, y_8, y_9\},$$
$$\{y_5, y_6, y_7, y_8, y_{10}\}, \{y_5, y_6, y_7, y_8, y_9, y_{10}\}, \{y_1, y_3, y_5, y_6, y_7, y_8\},$$
$$\{y_1, y_3, y_5, y_6, y_7, y_8, y_9\}, \{y_1, y_3, y_5, y_6, y_7, y_8, y_{10}\},$$
$$\{y_1, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}, \{y_1, y_3, y_5, y_6\}, \{y_1, y_5, y_7, y_8\},$$
$$\{y_1, y_5, y_7, y_8, y_9\}, \{y_1, y_5, y_7, y_8, y_{10}\}, \{y_1, y_5, y_7, y_8, y_9, y_{10}\},$$
$$\{y_1, y_5, y_6, y_7, y_8\}, \{y_1, y_5, y_6, y_7, y_8, y_9\}, \{y_1, y_5, y_6, y_7, y_8, y_{10}\}$$
$$\{y_1, y_5, y_6, y_7, y_8, y_9, y_{10}\}\}$$

$$\tau_l^{(\leq,CT)} = \{U, \phi, \{y_4\}, \{y_2, y_4\}, \{y_2, y_4, y_9\}, \{y_2, y_4, y_{10}\}, \{y_2, y_3, y_4\},$$
$$\{y_2, y_4, y_9, y_{10}\}, \{y_2, y_3, y_4, y_9\}, \{y_2, y_3, y_4, y_{10}\}, \{y_2, y_3, y_4, y_9, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4\}, \{y_1, y_2, y_3, y_4, y_9\}, \{y_1, y_2, y_3, y_4, y_{10}\}, \{y_2, y_3, y_4, y_6\},$$
$$\{y_1, y_2, y_3, y_4, y_9, y_{10}\}, \{y_2, y_3, y_4, y_6, y_9\}, \{y_2, y_3, y_4, y_6, y_{10}\},$$
$$\{y_2, y_3, y_4, y_6, y_9, y_{10}\}, \{y_1, y_2, y_3, y_4, y_6\}, \{y_1, y_2, y_3, y_4, y_6, y_9\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_{10}\}, \{y_1, y_2, y_3, y_4, y_6, y_9, y_{10}\}, \{y_1, y_2, y_3, y_4, y_7\},$$
$$\{y_1, y_2, y_3, y_4, y_7, y_9\}, \{y_1, y_2, y_3, y_4, y_7, y_{10}\}, \{y_1, y_2, y_3, y_4, y_7, y_9, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_7\}, \{y_1, y_2, y_3, y_4, y_6, y_7, y_9\}, \{y_1, y_2, y_3, y_4, y_6, y_7, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_7, y_9, y_{10}\}, \{y_1, y_2, y_3, y_4, y_6, y_7, y_8, y_9\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_7, y_9, y_{10}\}, \{y_1, y_2, y_3, y_4, y_6, y_7, y_8, y_9, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_9\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_{10}\}\}$$

$\tau_i^{(\leq,CT)} = P(U)$ (Discrete Topology):

$$\tau_u^{(\geq,CT)} = \{U, \phi, \{y_1, y_2, y_3, y_4, y_5, y_7, y_8\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_8\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_7, y_8\}, \{y_2, y_4, y_9\}, \{y_2, y_4, y_{10}\}, \{y_2, y_4\},$$
$$\{y_2, y_4, y_9, y_{10}\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_8, y_9\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_8, y_{10}\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_8, y_9, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_8\}, \{y_1, y_2, y_3, y_4, y_6, y_8, y_9\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_8, y_{10}\}, \{y_1, y_2, y_3, y_4, y_6, y_8, y_9, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_9\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_{10}\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7,\}, \{y_1, y_2, y_3, y_4, y_7, y_9, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4,, y_5, y_6, y_9\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4, y_5, y_6, y_9, y_{10}\}, \{y_1, y_2, y_3, y_4, y_6, y_7, y_8, y_9\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_7, y_8, y_{10}\}, \{y_1, y_2, y_3, y_4, y_6, y_7, y_8, y_9, y_{10}\},$$
$$\{y_1, y_2, y_3, y_4, y_6, y_7, y_9, y_{10}\}, \{y_1, y_2, y_3, y_4, y_6, y_7, y_8, y_{10}\}\}$$

Materials and Methods

This research will use qualitative methods with the object of research, where an analysis will be carried out and provide an interval ordered information system with mathematical tools such as topological generalizations, and control relations, covering approximations using Pawlak framework.

Research Stages

The research stages that will be used in this study can be seen with the following explanation:

1. **Research problem**. In the wording of the problem, it will be observed that the problems that occur are to be researched
2. **Study of literature**. A search for references such as books, research, journals, or websites related to the purpose of the paper will be followed and the collected reference information and data will take place by researchers to solve problems that have been worded earlier
3. **Data collection**. The data collection process that will be carried out by researchers is by collecting information make relationships and generating tools
4. **Enterprise analysis.** In the process of analyzing the mathematical tools, the research will illustrate data processing using IOISI Definition 4.1 approaching the Pawlak framework. The initial Phase will show the catalog of Pawlak rules that will be used in this work. Genializing tools will be clarified using Value String
5. **Conclusion**. By the end of the paper, conclusions will be set up from the final results of this research

Results and Discussion

The main aim of this paper is to generalate the Pawlak approximation space to a covering approximation space and then we use this result to work with the interval-ordered information systems. Results achieved by the proposed approach to generate two different rough approximations called *j*-lower and *j*-upper approximations. We applied *j*-rough notions such as *j*-rough membership, *j*-rough equality, and *j*-rough inclusion relations using topological generalizations. Our proposed method has raised the results handled from interval information systems. In this Research, there are many improved approaches to investing generalized approximation relative to a control relation in interval information systems, and we use the covering approximation in the rough set approach. This is a generalization of the Pawlak approach applied to interval information systems.

In our application example we succeeded to obtain data classification and data reductions using the order of its objectives. For example, Let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\}$ from Table 1, we have the following.

Example 6-1

Compute the after set and the fore set by the control relation $\Re_A^{\geq}$ in Table 1 considering Example 4-2.

$y_1\Re_A^{\geq} = \{y_1, y_5, y_7, y_8, y_9, y_{10}\}$,
$y_2\Re_A^{\geq} = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$
$y_3\Re_A^{\geq} = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$,
$y_4\Re_A^{\geq} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$
$y_5\Re_A^{\geq} = \{y_5, y_{10}\}, y_6\Re_A^{\geq} = \{y_5, y_6, y_8, y_9, y_{10}\}$,
$y_7\Re_A^{\geq} = \{y_5, y_7, y_8, y_9, y_{10}\} y_8\Re_A^{\geq} = \{y_5, y_8, y_9, y_{10}\}$,
$y_9\Re_A^{\geq} = \{y_{9,} y_{10}\}, y_{10}\Re_A^{\geq} = \{y_{10}\}$

and:

$y_1\Re_A^{\leq} = \{y_1, y_2, y_3, y_4\}, y_2\Re_A^{\leq} = \{y_2, y_3, y_4\}$
$y_3\Re_A^{\leq} = \{y_2, y_3, y_4\}, y_4\Re_A^{\leq} = \{y_4\}$
$y_5\Re_A^{\leq} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}, y_6\Re_A^{\leq} = \{y_2, y_3, y_4, y_6\}$
$y_7\Re_A^{\leq} = \{y_1, y_2, y_3, y_4, y_7\} y_8\Re_A^{\leq} = \{y_1, y_2, y_3, y_4, y_6, y_7, y_8\}$
$y_9\Re_A^{\leq} = \{y_{2,} y_4, y_9\}$
$y_{10}\Re_A^{\leq} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$

Example 6-2

The right covering ($r^{\geq}$-covering) and the left covering ($l^{\leq}$ -covering) by the control relation $\Re_A^{\geq}$ (resp. $\Re_A^{\leq}$)in Table 1 considering Example 4.2 are as in the following:

$$C_A^{\geq} = \{\{y_1, y_5, y_7, y_8, y_9, y_{10}\}, \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}, \{y_5, y_{10}\}\{y_5, y_6, y_8, y_9, y_{10}\}, \{y_5, y_7, y_8, y_9, y_{10}\}, \{y_5, y_8, y_9, y_{10}\}, \{y_9, y_{10}\}, \{y_{10}\}\}$$

and:

$$C_A^{\leq} = \{\{y_1, y_2, y_3, y_4\}, \{y_2, y_3, y_4\}, \{y_4\}, \{y_2, y_3, y_4, y_6\} \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}, \{y_1, y_2, y_3, y_4, y_7\}, \{y_2, y_{4,}, y_9\}, \{y_1, y_2, y_3, y_4, y_6, y_7, y_8,\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}\}$$

calculated as presented here: Let $Y = \{y_1, y_5, y_6, y_7, y_8\}$, then.

Clearly, $U = \bigcup_{y\in U} y\Re_A^{\geq}$ and $U = \bigcup_{y\in U} y\Re_A^{\leq}$.

$N_{(r\geq,A)}(y_1) = \{y_1, y_5, y_7, y_8, y_9, y_{10}\}$
$N_{(r\geq,A)}(y_2) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$
$N_{(r\geq,A)}(y_3) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$
$N_{(r\geq,A)}(y_4) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$
$N_{(r\geq,A)}(y_5) = \{y_5, y_{10}\}$
$N_{(r\geq,A)}(y_6) = \{y_5, y_6, y_8, y_9, y_{10}\}$
$N_{(r\geq A)}(y_7) = \{y_5, y_7, y_8, y_9, y_{10}\}$
$N_{(r\geq A)}(y_8) = \{y_5, y_8, y_9, y_{10}\}$
$N_{(r\geq A)}(y_9) = \{y_9, y_{10}\}, N_{(r\geq A)}(y_{10}) = \{y_{10}\}$

$N_{(l\leq,A)}(y_1) = \{y_1, y_2, y_3, y_4\}$
$N_{(l\leq,A)}(y_2) = \{y_2, y_3, y_4\}$
$N_{(l\leq,A)}(y_3) = \{y_2, y_3, y_4\}, N_{(l\leq A)}(y_4) = \{y_4\}$
$N_{(l\leq,A)}(y_5) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$
$N_{(l\leq,A)}(y_6) = \{y_2, y_3, y_4, y_6\}$
$N_{(l\leq,A)}(y_7) = \{y_1, y_2, y_3, y_4, y_7\}$
$N_{(l\leq,A)}(y_8) = \{y_1, y_2, y_3, y_4, y_6, y_7, y_8\}$
$N_{(l\leq,A)}(y_9) = \{y_2, y_4, y_9\}$
$N_{(l\leq,A)}(y_{10}) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$

$N_{(i\leq,A)}(y_1) = \{y_1\}, N_{(i\leq,A)}(y_2) = \{y_2, y_3\}$
$N_{(i\leq,A)}(y_3) = \{y_2, y_3\}, N_{(i\leq,A)}(y_4) = \{y_4\}$
$N_{(i\leq,A)}(y_5) = \{y_5\}, N_{(i\leq,A)}(y_6) = \{y_6\}$
$N_{(i\leq,A)}(y_7) = \{y_7\}, N_{(i\leq,A)}(y_8) = \{y_8\}$
$N_{(i\leq,A)}(y_9) = \{y_9\}, N_{(i\leq,A)}(y_{10}) = \{y_{10}\}$

and:

$N_{(u\geq,A)}(y_1) = \{y_1, y_2, y_3, y_4, y_5, y_7, y_8, y_9, y_{10}\}$,
$N_{(u\geq,A)}(y_2) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$,
$N_{(u\geq,A)}(y_3) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$,
$N_{(u\geq,A)}(y_4) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$
$N_{(u\geq,A)}(y_5) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_{10}\}$,
$N_{u\geq,A}(y_6) = \{y_2, y_3, y_4, y_5, y_6, y_8, y_9, y_{10}\}$,
$N_{(u\geq,A)}(y_7) = \{y_1, y_2, y_3, y_4, y_5, y_7, y_8, y_9, y_{10}\}$,
$N_{(u\geq,A)}(y_8) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$,
$N_{(u\geq,A)}(y_9) = \{y_2, y_4, y_9, y_{10}\}$,
$N_{(u\geq,A)}(y_{10}) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$

Example 6-3

The j-lower approximation and j-upper approximation of Y relative to the control relations $\Re_A^{\geq}$ and $\Re_A^{\leq}$ in Table 1 considering Example 4.2 can be:

$\underline{\Re}^{(\geq r,\, A)}(Y) = \phi$
$\overline{\Re}^{(\geq l,\, A)}(Y) \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$
$\underline{\Re}^{(\leq l,\, A)}(Y) = \phi, \overline{\Re}^{(\leq l,\, A)}(Y) = \{y_1, y_5, y_6, y_7, y_8\}$
$\underline{\Re}^{(\leq i,\, A)}(Y) = \{y_1, y_5, y_6, y_8\}$,
$\overline{\Re}^{(\leq i,\, A)}(Y) = \{y_1, y_5, y_6, y_8\}, \underline{\Re}^{(\geq u,\, A)}(Y) = \phi$,
$\overline{\Re}^{(\geq u,\, A)}(Y) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_{10}\}$

The right j-accuracy measure and the left j-accuracy measure of Y relative to the control relation $(\Re_j)_A^{\geq}$ and $(\Re_j)_A^{\leq}$ can be calculated as presented here:

$$\alpha((\Re_r)_A^{\geq}, Y) = \frac{|(\underline{\Re}_r)_A^{\geq}(Y)|}{|(\bar{\Re}_r)_A^{\geq}(Y)|} = 0$$

$$\alpha((\Re_l)_A^{\leq}, Y) = \frac{|(\underline{\Re}_l)_A^{\leq}(Y)|}{|(\bar{\Re}_l)_A^{\leq}(Y)|} = 0$$

$$\alpha((\Re_i)_A^{\leq}, Y) = \frac{|(\underline{\Re}_i)_A^{\leq}(Y)|}{|(\bar{\Re}_l)_A^{\leq}(Y)|} = 1$$

$$\alpha((\Re_u)_A^{\geq}, Y) = \frac{|(\underline{\Re}_u)_A^{\geq}(Y)|}{|(\bar{\Re}_u)_A^{\geq}(Y)|} = 0$$

Note that from Example 5.16 and the above:

$$\alpha\left((\Re_j)_{CT}^{\geq}, Y\right) \geq \alpha\left((\Re_j)_A^{\geq}, Y\right)$$

where, $j \in \{r, u\}$.

$$\alpha\left((\Re_j)_{CT}^{\leq}, Y\right) \geq \alpha\left((\Re_j)_A^{\leq}, Y\right)$$

where $j \in \{l, i\}$

Conclusion and Future Works

In summary, the theory of rough set has been trusted to be a valuable mathematical procedure for ranking and prognosis, and the one generalization of the traditional accession of the rough set is the control-based approach of the rough set, which is fundamentally dependent on substitution of the indistinguishability relation by a control relation. Interval systems of information are a significant kind of data tables, that are indistinguishable models of information systems of one value. According to this study, we can say that the use of topological generalizations has improved the results obtained from interval information systems. In addition, the disruption of a general relation from which a double topological base is generated supports decision-making so that the decision is not isolated to a single topological and is settled equally between the two topologies, and one can observe that this approach can help in data classification and data reductions using the order of its objectives The future research in this direction suggests simplifications on these generalizations by making new algorithms, which simplify the calculations on it. Furthermore, we can add more topologies to support decision-making in real-life applications.

Novelty Statement

The research is recent and not derived from scientific theses and it did not use data from undocumented sources since we generalize classification approximation spaces to new spaces using topology. The research problem here is how to take an accurate decision from the available data of ordered information systems counting on the possibility to assort objects of this system using topological tools and make use of them in applications. Rough sets play a crucial role in decision-making in an information system via new topology.

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Author’s Contributions

Nadia El Mokhtar Gheith: Prepared the research in the of written, drawn conclusions and proved them, as well as maked examples.

A. S. Salama: Contribute to the idea and drafted methods.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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