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The Algebraic K-Theory of Finitely Generated Projective Supermodules P(R) Over a Supercommutative Super-Ring R

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Abstract: Problem statement: Algebraic K-theory of projective modules over commutative rings were introduced by Bass and central simple superalgebras, supercommutative super-rings were introduced by many researchers such as Knus, Racine and Zelmanov. In this research, we classified the projective supermodules over (torsion free) supercommutative super-rings and through out this study we forced our selves to generalize the algebraic K-theory of projective supermodules over (torsion free) supercommutative super-rings. **Approach:** We generalized the algebraic K-theory of projective modules to the super-case over (torsion free) supercommutative super-rings. **Results:** we extended two results proved by Saltman to the supercase. **Conclusion:** The extending two results, which were proved by Saltman, to the supercase and the algebraic K-theory of projective supermodules over (torsion free) supercommutative super-rings would help any researcher to classify further properties about projective supermodules.

Key words: Projective supermodules, superinvolutions, brauer groups, brauer-wall groups

INTRODUCTION

An associative super-ring $R = R_0 + R_1$ is nothing but a $\mathbb{Z}_2$-graded associative ring. A $\mathbb{Z}_2$-graded ideal $I = I_0+I_1$ of an associative super-ring is called a superideal of R. An associative super-ring R is simple if it has no non-trivial superideals. Let R be an associative super-ring with $1 \in R_0$ then R is said to be a division super-ring if all nonzero homogeneous elements are invertible, i.e., every $0 \neq r_\alpha \in R_\alpha$ has an inverse r_α^{-1}, necessarily in R_a. If $R = R_0 + R_1$ is an associative super-ring, a (right) R-supermodule M is a right R-module with a grading $M = M_0 + M_1$ as R_0-modules such that $m_\alpha r_\beta \in M_{\alpha+\beta}$ for any $m_\alpha \in M_\alpha$, $r_\beta \in R_\beta$, $\alpha,\beta \in \mathbb{Z}_2$. An R-supermodule M is simple if $MR \neq \{0\}$ and M has no proper subsupermodule. Following[4] we have the following definition of R-supermodule homomorphism. Suppose M and N are R -supermodules. An R-supermodule homomorphism from M into N is an R_0-module homomorphism $h_\gamma : M \to N$, $\gamma \in \mathbb{Z}_2$, such that $M_\alpha h_\gamma \subseteq N_{\alpha+\gamma}$. Let K be a field of characteristic not 2. An associative superalgebra is a $\mathbb{Z}_2$-graded associative K-algebra $A = A_0 + A_1$. A superalgebra A is central simple over K, if $\hat{Z}(A) = K$, where $(\hat{Z}(A))_\alpha = \{\alpha_\alpha \in A_\alpha : \alpha_\alpha b_\beta = (-1)^{\alpha\beta} b_\beta \alpha_\alpha \forall \beta_\beta \in A_\beta\}$ and the only superideals of A are (0) and A. Through out this study we let R be a supercommutative super-ring ($\hat{Z}(A) = R$) with $1 \in R_0$. An R-superalgebra $A = A_0 + A_1$ is called projective R-supermodule if it is projective as a module over R. Define the superalgebra $A^e = A^o \hat{\otimes}_R A$, then A is right A^e-supermodule. There is a natural map π from A^e to A given by deleting o,s and multiplying.

In[2], Childs, Garfinkel and Orzech proved some results about finitely generated projective supermodules over R, where R is a commutative ring. In[1], we generalized the same results about finitely generated projective supermodules over R, where R is a supercommutative super-ring. Here are the results:

Proposition 1: Let M be an R-supermodule and A an R-superalgebra then there exist isomorphisms of R-superalgebras:

$$A \hat{\otimes}_R End_R(M) \cong End_R(M) \hat{\otimes}_R A$$

Corollary 1: Let P and Q be a finitely generated projective supermodules over R, then:

$$End_R(P) \hat{\otimes}_R End_R(Q) \cong End_R(Q) \hat{\otimes}_R End_R(P) \cong End_R(P \hat{\otimes}_R Q)$$

Theorem 1: Let A be an R-superalgebra. The following conditions are equivalent:

- A is projective right A^e-supermodule
- $0 \mapsto \ker(\pi) \mapsto A^e \xrightarrow{\pi} A \to 0$ splits as a sequence of right A^e-supermodules

- $(A^e)_0$ contains an element ε such that $\pi(\varepsilon)=1$ and $\varepsilon(1\otimes a_\alpha)=\varepsilon(a_\alpha\otimes 1)$ for all $a_\alpha \in A_\alpha$

Definition 1: We say that A is R-separable if conditions (1-3) above hold.

Remarks:

- Condition (3) states that A is R-separable if and only if it is R-separable of the sense of ungraded algebras
- It is easy to see that ε defined above is idempotent. A is a central separable R-superalgebra if it is separable as an R-algebra, thus our Azumaya R-algebras A are those separable R-algebras which are superalgebras over R and whose supercenter is R

For any R-superalgebra A we have seen that A is naturally a right A^e -supermodule. This induces an R-superalgebra homomorphism μ from A^e to End_R (A) by associating to any element $x_\alpha \otimes y_\beta$ of A^e the element $x_\alpha y_\beta$ where for any $a_\gamma \in A_\gamma$:

$$a_\gamma \mu(x_\alpha \otimes y_\beta) = a_\gamma.(x_\alpha y_\beta) = (-1)^{\alpha\gamma} x_\alpha a_\gamma y_\beta$$

Theorem 2: Let A be an R-superalgebra. The following conditions are equivalent:

- A is an Azumaya R-superalgebra
- A is finitely generated faithful projective R-supermodule and μ is an isomorphism

MATERIALS AND METHODS

Suppose C is any category and obj(C) the class of all objects of C and let C(A,B) be the set of all morphisms A→B, where A,B $\in$obj(C). A groupoid is a category in which all morphisms are isomorphisms.

Definition 2: A category with product is a groupoid C, together with a product functor $\perp$: C×C→C which is assumed to be associative and commutative.

A functor $F:(C,\perp)\to(C',\perp')$ of categories with product is a functor $F:C\to C'$ which preserves the product.

Examples:

- Let R be any supercommutative super-ring and let P(R) denote the category of finitely generated projective supermodules over R with isomorphisms as morphisms. It is a category with product if we set $\perp = \oplus$
- The subcategory FP(R) of P(R) with finitely generated faithful projective supermodules as objects. Her we set $\perp = \hat{\otimes}_R$
- The category Az(R) of Azumaya superalgebras over R. Her we take $\perp = \hat{\otimes}_R$

If C(R) denotes one of the categories mentioned above and if R→R′ is a homomorphism of super-rings. Then $R'\hat{\otimes}_R$ induces a functor $C(R)\to C'(R')$ preserving product.

Definition 3: Let C be a category with product. The Grothendieck group of C is defined to be an abelian group K_0 C, together with the map $(\ )_C$: obj(C) →K_0C, which is universal for maps into abelian groups satisfying:

- if $A \cong B$, then $(A)_C = (B)_C$
- $(A \perp B)_C = (A)_C + (B)_C$

Definition 4: A composition on a category $(C,\perp)$ is a composition of objects of C, which satisfies the following condition: if $A\circ A'$ and $B\circ B'$ are defined then so also is $(A\perp B)\circ(A'\perp B')$ and:

$$(A\perp B)\circ(A'\perp B') = (A\circ A')\perp(B\circ B')$$

Definition 5: If $(C,\perp,\circ)$ is a category with product and composition. Then the Grothendieck group of C is defined to be an abelian group K_0 C, together with a map:

$$(\ )_C : obj(C) \to K_0C$$

which is universal for maps into abelian groups satisfying the two conditions in Definition 3 and:

If $A\circ B$ is defined, then $(A\circ B)_C = (A)_C + (B)_C$

An easy computation gives us the following result.

Proposition 2: Let $(C,\perp,\circ)$ be a category with product and composition. Then:

- Every element of K_0 C has the form $(A)_C$-$(B)_C$ for some A, B in obj(C)
- $(A)_C = (B)_C$ if and only if $\exists C, D_0, D_1, E_0, E_1 \in$ obj(C), such that $D_0 \circ D_1$ and $E_0 \circ E_1$ are defined and $A\perp C\perp(D_0\circ D_1)\perp E_0\perp E_1 \cong B\perp C\perp D_0\perp D_1\perp(E_0\circ E_1)$

- If F: $C\to C'$ is a functor of categories with product and composition, then F preserves the composition. Moreover, the map $K_0F: K_0C\to K_0C'$ given by $(A)_C \to (FA)_{C'}$ is well-defined and makes K_0F a functor into abelian groups

Now let $(C,\perp)$ be a groupoid. For $A\in obj(C)$, we write G (A) = C(A,A), the group of automorphisms of A. If f: A $\to$ B is an isomorphism, then we have a homomorphism G(f): F(A) $\to$ G(B), given by $G(f)(\alpha) = f\alpha f^{-1}$.

We shall construct, out of C, a new category ΩC. we take obj(ΩC) to be the collection of all automorphisms of C. If $\alpha \in obj(\Omega C)$ is an automorphism of $A\in C$, we write (A,α) instead of α. A morphism $(A,\alpha) \to (B,\beta)$ in ΩC is a morphism $f: A\to B$ in C such that the diagram in Fig. 1 is commutative, that is G(f) (α) = β. The product in ΩC is defined by setting $(A, \alpha) \perp (\beta, \beta) = (A\perp B, \alpha \perp \beta)$. The natural composition $\circ$ is defined in ΩC as follows: if $\alpha, \beta \in obj(\Omega C)$ are automorphisms of the same object in C, then $\alpha\circ\beta = \alpha\ \beta$ and:

$$(\alpha\perp \beta)\circ(\alpha'\perp \beta') = \alpha\alpha' \perp \beta\beta'$$

Definition 6: If $(C, \perp)$ is a category with product, we define:

$$K_1C = K_0\Omega C$$

If F: $C\to C'$ is a functor, then $\Omega F: \Omega C\to\Omega C'$, preserving product and composition, so we obtain homomorphisms $K_iF: K_iC \to K_iC'$, i = 0,1.

If P(R) is the category of finitely generated projective R-supermodules, where R is a supercommutative super-ring and their isomorphisms with $\oplus$. Then the tensor product $\hat{\otimes}_R$ is additive with respect to $\oplus$ so that it induces on K_0 P(R) a structure of commutative ring.

The next following results are just the generalizing of the results proved by H. Bass to the supercase.

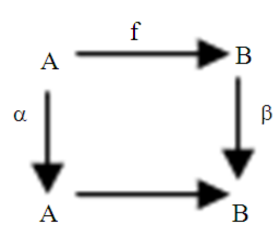

Fig. 1: Set of morphisms

If $Z \in spec(R)$ (i.e., $Z\subseteq R$ is a prime superideal) and $P\in P(R)$, then P_Z is a free R_Z -supermodule and its rank is denoted by rk_P (Z). The map:

$$rk_P: spec(R)\to\mathbb{Z}$$

given by $Z\to rk_P(Z)$ is continuous and is called the rank of P. As R is a supercommutative super-ring, K_0 P(R) and $Q\hat{\otimes}_R K_0P(R) = QK_0P(R)$ are rings with multiplication induced by $\hat{\otimes}_R$. Since:

$$rk_{P\oplus Q} = rk_P + rk_Q$$

and

$$rk_{P\oplus_R Q} = rk_P rk_Q$$

We have a rank homomorphism:

$$rk_P: K_0P(R)\to C$$

where C is the ring of continuous functions $spec(R)\to\mathbb{Z}$.

The rank homomorphism rk is splitting by the ring homomorphism $C\to K_0 P(R)$, so that:

$$K_0 P(R) \cong C \oplus\widetilde{K_0}P(R)$$

where, $\widetilde{K_0}P(R) = ker(rk)$ So:

$$\mathbb{Q}\otimes_{\mathbb{Z}} K_0P(R) \cong (\mathbb{Q}\otimes_{\mathbb{Z}} C)\oplus(\mathbb{Q}\otimes_{\mathbb{Z}} \widetilde{K_0}P(R))$$

The next results generalize the results proved by H. Bass.

Theorem 3: Suppose max(R), the space of maximal superideals of R, is noetherian space of dimension d, then:

- If $x\in K_0P(R)$ and $rk(x)\ge d$, then $x = (p)_{P(R)}$ for some $P\in P(R)$
- If $rk((P)_{P(R)}) > d$ and if $(P_{P(R)}) = (Q_{P(R)})$, then $P\approx Q$
- $(\widetilde{K_0}P(R))^{d+1} = 0$

Proposition 3: The following conditions on R-supermodule P are equivalent:

- P is a finitely generated projective supermodule over R and has zero ahnihlator
- $P\in P(R)$ and has every where positive rank

- $\exists$ a supermodule Q and a positive integer n such that $P\hat{\otimes}_R Q \approx R^n$

RESULTS AND DISCUSSION

Let P(R) be the category of finitely generated projective supermodules over R, Az(R) the category of Azumaya superalgebras over R and Prog(R) the category of finitely generated faithful projective R-supermodules.

A useful fact to be remember is that since R is supercommutative super-ring, $P\in$Prog(R) if and only if $P\in$Prog(R) and P is faithful. If $A,B\in$Az(R) are equivalent in BW(R) (the Brauer-Wall group of R), we will write $A \sim B$. If M is a supermodule over R, then nM is the n-fold direct sum of M. If $P\in$ P(R) let (P) be the image of P in K_0 P(R) and {P} in $\mathbb{Q}\otimes_{\mathbb{Z}} K_0 P(R) = \mathbb{Q}$ K_0 P(R). The next results generalize the results proved by[6].

Theorem 4: Let $P, P', Q \in$ P(R). Then:

- $P\in$Prog(R) if and only if there is a Q in P(R) such that $P\hat{\otimes}_R Q$ is free
- If $x\in \mathbb{Q}$ K_0P(R) and rk(x) > 0 then $x = \left(\frac{1}{m}\right)\{Q\}$ for some $Q\in$ Prog(R), $m>0$ an integer
- If $\{P\}=\{Q\}$, $P\in$Prog(R), then there is an integer $n>0$ such that $nP \approx nQ$
- If $Q\in$ Prog(R) and $((P)-(P'))(Q)=0$ then there is an integer n > 0 such that $nP \approx nP'$
- If $P\in$Prog(R) and rk_P is a square then there is an integer n > 0 and $Q\in$ Prog(R) such that $n^2P \approx Q\hat{\otimes}_R Q$

Let R/S be Galois extension of supercommutative super-rings with finite Galois Group G. $M = M_0 + M_1$, an R-supermodule, has a G-action if there is a group injection $\varphi: G \to \text{Aut}(M)$ such that $\varphi(\sigma)$ is σ-linear for all $\sigma\in G$. That is, $\varphi(\sigma)(m_\alpha r_\beta) = \varphi(\sigma)(m_\alpha)\sigma(r_\beta)$. Let $M^G = \{m \in M : \varphi(\sigma)(m) = m \text{ for all } \sigma \in G\}$. The following fact was proved in[1], if $M\in$ Prog(R), M^G Prog(S) then:

$$R\hat{\otimes}_S M^G \cong M$$

Again let R/S be a Galois extension of super-commutative super-rings with Galois group $G=\{1,\sigma\}$. Let A be any central separable R-superalgebra, we define A^σ as follows, set $A^\sigma = A$ as a super-ring, but the product by a scalar. on A^σ is defined by $\lambda a = \sigma(\lambda)a$ for all $\lambda\in R$. Then one easily check that A^σ is a central separable R- superalgebra.

Now let $\tau: A^\sigma\hat{\otimes}_R A \to A^\sigma\hat{\otimes}_R A$, be defined by $\tau(a_\alpha \otimes b_\beta) = (-1)^{\alpha\beta} b_\beta \otimes a_\alpha$, then τ is a σ-linear automorphism. In particular τ is S-linear. Define the Corestriction:

$$\text{Tr}(A) = \{ x \in A^\sigma\hat{\otimes}_R A \mid \tau(x) = x \}$$

Obviously, Tr(A) is an S-superalgebra. But by[3]Tr(A) is an S-progenerator as an S-supermodule, if A is an R-progenerator as an R-supermodule. Moreover if A is central separable over R then by[3] Tr(A) is central separable over S.

Lemma 1: Let R/S be a Galois extension of super-commutative super-rings with Galois group $G=\{1,\sigma\}$. Let A, B be R-supermodules (superalgebras), $P\in$ Prog(R):

- If A and B have G-action, so does $M = A\hat{\otimes}_R B$ and $M^G = A^G \otimes_R B^G$
- $\text{Tr}(A\hat{\otimes}_R B) \cong \text{Tr}(A) \hat{\otimes}_S \text{Tr}(B)$
- If E = $\text{End}_R(P)$, $\text{Tr}(E) \cong \text{End}_S(\text{Tr}(P))$

Theorem 5: Let $A\in$ Az(R) and $P,Q\in$ Prog(A) such that $P\approx Q$ as R-supermodules. Then there is an integer n > 0 such that $nP \approx nQ$ as A-supermodules.

Proof: $A\hat{\otimes}_R \text{End}_A(P) \cong \text{End}_R(P) \cong \text{End}_R(Q) \cong A\hat{\otimes}_R \text{End}_A(Q)$. Tensoring by A° yields that:

$$\text{End}_R(A)\otimes_R \text{End}_A(P) \cong \text{End}_R(A)\otimes_R \text{End}_A(Q)$$

or

$$\text{End}_A(A\otimes_R P) \cong \text{End}_A(A\otimes_R Q)$$

where, A acts on $A\hat{\otimes}_R P$ ($A\hat{\otimes}_R Q$) by acting on P (Q). Using[3], There is a rank one projective R-supermodule I, such that $A\hat{\otimes}_R P \cong A\hat{\otimes}_R Q \hat{\otimes}_R I$ as A-supermodules. Theorem 4(a) implies that $mR\hat{\otimes}_R P \cong mR\hat{\otimes}_R Q \hat{\otimes}_R I$ as A-supermodules, for some m > 0 and $m'R \cong m'R\hat{\otimes}_R I$ as R-supermodules, for some different m′. Finally, n = mm′ will satisfy the theorem.

On a superalgebra A, a map $J: A \to A$ is called a superinvolution if J^2 is the identity and J is an antiautomorphism. More explicitly, $(a_\alpha)^{J^2} = a_\alpha$, $(a_\alpha + b_\beta)^J = a_\alpha^J + b_\beta^J$ and $(a_\alpha b_\beta)^J = (-1)^{\alpha\beta} b_\beta^J a_\alpha^J$ for all $a_\alpha, b_\beta \in A$. Let $C = \hat{Z}(A)$ (the super-center of A) then J must preserve C. If J is the identity on C, J is a superinvolution of the first kind. If not, J induces an automorphism of C of order 2 and J is said to be of the second kind. Two superinvolutions J, J′ which agree on C are said to be of the same kind.

The following theorem generalizes of[6].

Theorem 6: If $A \in Az(R)$ and $A \hat{\otimes}_R A \sim 1$, then there is a $B \in Az(R)$, such that $B \sim A$ and $B \cong B^\circ$.

Another way of viewing an isomorphism $B \cong B^\circ$ is that B has an antiautomorphism, J, of the first kind. Now, we are ready to prove the following result.

Theorem 7: Suppose A is a super-ring with antiautomorphism J such that J^2 is inner, induced by a $w_0 \in A_0$ such that $w_0(w_0)^J = (w_0)^J w_0 = 1$. Then $M_2(A)$ has a superinvolution of the same kind.

Proof Let L be the inverse map to J. Since:

$$w_0^{-1} a_\alpha w_0 = (a_\alpha)^{J^2}$$

We have $(a_\alpha)^J (w_0)^J = (w_0)^J (a_\alpha)^L$ and $(a_\alpha)^L w_0 = w_0 (a_\alpha)^J$, so the map:

$$\begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \mapsto \begin{pmatrix} (d_\alpha)^J & (w_0)^J (b_\alpha)^L \\ w_0 (c_\alpha)^J & (a_\alpha)^L \end{pmatrix}$$

Is a superinvolution on M_2A of the same kind of J.

Next we try to find the conditions on a central separable R-superalgebra A to have a superinvolution of the second kind, if R is a connected super-ring. In the next theorem we try to find the conditions on A= $End_R(P)$ to have a superinvolution of any kind, where P is an R-progenerator as a supermodule over R, if R is a connected super-ring.

The following theorem involves assuming that R, the base super-ring, is semilocal. We will use the fact, from[5], that if A, B are central separable R-algebras, $A \sim B$ and the rank of A equals rank of B, then $A \cong B$, which is also true in the superalgebra case (i.e., if A, B are central separable R-superalgebras, $A \sim B$ and the rank of A equals rank of B, then $A \cong B$). Let M be the Jacobson radical of R. Then $\overline{A} = A / MA$ is a finite direct sum of simple superalgebras. We call $\overline{A}$ is perfect if every simple subsuperalgebra of $\overline{A}$ admits a superinvolution of the second kind.

Theorem 8: Suppose R is a connected semilocal super-ring and A is a central separable R-superalgebra. Suppose R/S is a Galois extension with Galois group $\{1, \sigma\}$. Then A has a superinvolution of the second kind extending σ if and only if $Tr(A) \sim 1$ and $\overline{A}$ is perfect.

Proof: Suppose A has a superinvolution, *, extending σ. Then it is easy to Check that $\overline{A}$ is perfect. Also * induces an isomorphism $A^\sigma \cong A^\circ$, so there is an isomorphism:

$$\varphi : A^\sigma \hat{\otimes}_R A \to End_R(A)$$

given by $x_\gamma (a_\alpha \otimes b_\beta)^\varphi = (-1)^{\alpha\gamma} a_\alpha^* x_\gamma b_\beta$. Set $A' = A'_0 + A'_1$, where $A'_\alpha = \{a_\alpha \in A_\alpha : a_\alpha^* = a_\alpha\}$.

Since * is σ-linear R-supermodule automorphism of A, the S-supermodule A' is an S progenerator as a module over S. φ induces an isomorphism $Tr(A) \cong End_S(A')$, hence $Tr(A) \sim 1$.

Conversely, since R is a connected semilocal super-ring, one easily sees that S is a connected semilocal super-ring also. Let Tr(A) $\cong End_S(P)$. In other words let $\tau : A^\sigma \hat{\otimes}_R A \to A^\sigma \hat{\otimes}_R A$ given by $(a_\alpha \otimes b_\beta)^\tau = (-1)^{\alpha\beta} b_\beta \otimes a_\alpha$, be a σ-linear automorphism. Then Tr(A) is the fixed super-ring of $A^\sigma \hat{\otimes}_R A$ under τ. Say Tr(A) $\cong End_S(P)$, where Pis an S-progenerator as a supermodule over S. Then $A^\sigma \hat{\otimes}_R A \cong R \hat{\otimes}_R End_S(P) \cong End_R(R \hat{\otimes}_S P)$ and if $\varphi = \sigma \otimes 1 : R \hat{\otimes}_S P \to R \hat{\otimes}_S P$, then $(x_\gamma (a_\alpha \otimes b_\beta))^\varphi = x_\gamma^\varphi (a_\alpha \otimes b_\beta)^\tau$, for all $x_\gamma \in R \hat{\otimes}_S P$ and $a_\alpha \otimes b_\beta \in A^\sigma \hat{\otimes}_R A$. Since R is connected, $rank_R(A) = rank_R(A^\sigma)$, but:

$$A^\sigma \hat{\otimes}_R A \cong End_R(R \hat{\otimes}_S P)$$

Therefore:

$$A^\sigma \hat{\otimes}_R (A \hat{\otimes}_R A^\circ) \cong A^\sigma \hat{\otimes}_R End_R(A) \cong End_R(R \hat{\otimes}_S P) \hat{\otimes}_R A^\circ$$

So by[5], $A^\sigma \cong A^\circ$, which implies that $End_R(A) \cong End_R(R \hat{\otimes}_S P)$, but the R-rank of A equals the

R-rank of $R\hat{\otimes}_S P$. So again by[5], $A \cong R\hat{\otimes}_S P$. In other words, A has a σ -linear antiautomorphism J such that for all $a_\alpha, x_\gamma, b_\beta$ in A, setting $x_\gamma(a_\alpha \otimes b_\beta) = (-1)^{\alpha\lambda} a_\alpha^J x_\gamma b_\beta$ yields the isomorphism $A^\sigma \otimes_R A \cong End_R(A)$ and the map $\varphi : \sigma \otimes 1 : A(\cong R\hat{\otimes}_S P) \to A$ satisfies $\varphi^2 = 1$ and $(x_\gamma.(a_\alpha \otimes b_\beta))^\varphi = x_\gamma^\varphi.(a_\alpha \otimes b_\beta)^\tau$. Therefore:

$$(-1)^{\alpha\lambda}(a_\alpha^J x_\gamma b_\beta)^\varphi = (-1)^{\alpha\beta} x_\gamma^\varphi.(b_\beta \otimes a_\alpha) = (-1)^{\beta(\alpha+\gamma)} b_\beta^J x_\gamma^\varphi a_\alpha$$

(φ respects the grading). For $w = 1^\varphi \in A_0$ we have $ww^J = w^J w = 1$ and $wa_\alpha w^{-1} = a_\alpha^{J^2}$ and:

$$\varphi^2 = 1,\ (a_\alpha^J x_\gamma b_\beta)^\varphi = (-1)^{\alpha\lambda}(-1)^{\beta(\alpha+\gamma)} b_\beta^J x_\gamma^\varphi a_\alpha \qquad (1)$$

Lemma 2: Let A be a central separable R-superalgebra, with J and φ satisfying (1). Then A has a superinvolution agreeing with J on R if φ fixes a unit of A_a.

Proof: If u_a is a unit in A_a such that $u_\alpha^\varphi = u_\alpha$ then $u_\alpha = (1.u_\alpha)^\varphi = u_\alpha^J w$, so $(u_\alpha^J)^{-1} u_\alpha = w$, but $(u_\alpha^J)^{-1} = (-1)^\alpha (u_\alpha^{-1})^J$, therefore $w = (-1)^\alpha (u_\alpha^{-1})^J u_\alpha$, implying that $x_\gamma^{J'} = u_\alpha^{-1} x_\gamma^J u_\alpha$ is a superinvolution since J' is an antiautomorphism on A and:

$$\begin{aligned}(x_\gamma^{J'})^{J'} &= u_\alpha^{-1}(u_\alpha^{-1} x_\gamma^J u_\alpha)^J u_\alpha = (-1)^\alpha u_\alpha^{-1}(u_\alpha^J x_\gamma^{J^2} (u_\alpha^{-1})^J) u_\alpha \\ &= u_\alpha^{-1} u_\alpha^J (w x_\gamma w^{-1}) w \\ &= x_\gamma,\ \text{since } u_\alpha^{-1} u_\alpha^J = w^{-1}\end{aligned}$$

Continuing proof of the theorem: Let M be the jacobson radical of R. Then $\bar{A} = A/MA$ is a finite direct sum of simple superalgebras. On $\bar{A}$, φ and J induce maps $\bar{\varphi}$ and $\bar{J}$ satisfying (1). Every preimage of a unit $\bar{u}_\alpha$ of $\bar{A}$ is a unit u_a of A_a. Thus we can change J by conjugation with a unit u_a, to make $\bar{J}$ any desired antiautomorphism of $\bar{A}$ of the same kind. In fact, if J' is defined by $x_\gamma^{J'} = u_\alpha^{-1} x_\gamma^J u_\alpha$, we can find a corresponding φ' so that J', φ' satisfy (1). Specifically if L is the inverse map to J, we can set $x_\gamma^{\varphi'} = u_\alpha^{-1} x_\gamma^\varphi u_\alpha^L$, to show that we have:

$$\begin{aligned}(x_\gamma^{\varphi'})^{\varphi'} &= u_\alpha^{-1}(u_\alpha^{-1} x_\gamma^\varphi u_\alpha^L)^\varphi u_\alpha^L \\ &= (-1)^\alpha u_\alpha^{-1}(u_\alpha^{LJ} x_\gamma z_\alpha) u_\alpha^L\end{aligned}$$

where $z_\alpha^J = u_\alpha^{-1}$ and hence $z_\alpha = z_\alpha^{JL} = (u_\alpha^{-1})^L$, so that $(x_\gamma^{\varphi'})^{\varphi'} = (-1)^\alpha x_\gamma (u_\alpha^{-1})^L u_\alpha^L = x_\gamma$ since $(-1)^\alpha (u_\alpha^L)^{-1} = (u_\alpha^{-1})^L$. It suffices to find $\bar{u}_\alpha$ of $\bar{A}$ such that $(\bar{u}_\alpha)^\varphi + \bar{u}_\alpha$ is a unit, for if u_α is a preimage of $\bar{u}_\alpha$, then $(u_\alpha)^\varphi + u_\alpha$ will be a φ fixed unit of A_a. Since $\bar{A}$ is perfect, it suffices to prove.

Lemma 3: Let $\bar{A}$ be a finite dimensional central simple superalgebra over a field F with a superinvolution J of the second kind and any associated φ to J then there is an element $\bar{a}_\alpha$ in $\bar{A}_\alpha$ such that $(\bar{a}_\alpha)^\varphi + \bar{a}_\alpha$ is a unit.

Proof: The element $w = 1^\varphi$ is central since J is a superinvolution. If $w \neq -1$, then $\bar{a}_0 = 1$ will do. If $w = -1$, then $(\bar{a}_\alpha)^\varphi = (\bar{a}_\alpha)^J w = -(\bar{a}_\alpha)^J$. Since J is of order 2 on F, there is f in F such that $f - f^J \neq 0$, so again take $\bar{a}_0 = f - f^J$.

Lemma 4: Suppose Q is a right $A^e = A^\circ \hat{\otimes}_R A$ -supermodule, then:

$$Q = M \oplus I$$

where, M is the R-subsupermodule of Q generated by all elements of the form $(a_\alpha \otimes 1 - 1 \otimes a_\alpha) q_\beta$, where $a_\alpha \in A_\alpha$ and $q_\beta \in Q_\beta$. If Q is R-projective as a supermodule over R then:

$$\text{rank}_R(A).\text{rank}_R(I) = \text{rank}_R(Q).$$

Proof: Consider the well-known split exact sequence of A^e-supermodules:

$$0 \to J \to A^e \xrightarrow{\mu} A \to 0$$

where, $\mu(a_\alpha \otimes b_\beta) = a_\alpha b_\beta$ and J is a right super-ideal of A^e generated by all elements of the form $a_\alpha \otimes 1 - 1 \otimes a_\alpha$ where $a_\alpha \in A_\alpha$. Suppose Q is a right A^e-supermodule. Tensoring by Q over A^e yields a split exact sequence of R-supermodules:

$$0 \to Q\hat{\otimes}_{A^e} J \to Q\hat{\otimes}_{A^e} A^e \xrightarrow{1\otimes\mu} Q\hat{\otimes}_{A^e} A \to 0$$

of course, $Q\hat{\otimes}_{A^e} A^e \cong Q$ under the map $a_\alpha \otimes z_\beta \mapsto a_\alpha z_\beta$. Under this isomorphism $Q\hat{\otimes}_{A^e} J$ is mapped onto M defined above. Thus $Q \cong M \oplus I$, where $I \cong Q\hat{\otimes}_{A^e} A$. But:

$$I\hat{\otimes}_R A \cong Q\hat{\otimes}_{A^e}(A\hat{\otimes}_R A) \cong Q\hat{\otimes}_{A^e}(A^\circ\hat{\otimes}_R A)$$

as an R-supermodules, therefore, $I\hat{\otimes}_R A \cong Q\hat{\otimes}_{A^e} A^e \cong Q$.

Suppose R is a local supercommutative super-ring, σ an automorphism of R of order 2, P is an R-progenerator as a supermodule over R and I a rank one R-projective supermodule. A morphism $e: P\hat{\otimes}_R P \to I$ is called a bilinear I form on P, a morphism $e: P^\sigma\hat{\otimes}_R P \to I$ is called a σ bilinear I form on P. The image $e(p_\alpha \otimes q_\beta)$ is often written as $e(p_\alpha, q_\beta)$ and in either case, e can be thought of as a map $e: P\times P \to I$. Such a form induces a map $e^*: P \to Hom_R(P,I)$ $(P^\sigma \to Hom_R(P,I))$ given by $e^*(p_\alpha)(q_\beta) = e(p_\alpha, q_\beta)$. In a similar manner, we define $e_*: P \to Hom_R(P,I)$ $(P \to Hom_R(P^\sigma,I))$ given by $e_*(p_\alpha)(q_\beta) = e(p_\alpha, q_\beta)$. If e^* and e_* are isomorphisms then we say e is nondegenerate. The next final result shows that the existence of superinvolutions on $End_R(p)$, where $End_R(p)$ is an R-progenerator as a supermodule over R, is equivalent to the existence of forms on P and this result was proved in[1].

Theorem 9: Let R be a connected super-ring and A = $End_R(p)$ be a central separable R-superalgebra such that A is an R-progenerator as a supermodule over R, then:

- A has a superinvolution of the first kind if and only if there is a rank one R-projective I, a nondegenerate bilinear I form e on P and a $\delta \in R_0$ such that $\delta^2 = 1$ and $e(x_\alpha, y_\beta) = (-1)^{\alpha\beta}\delta e(y_\beta, x_\alpha)$ for all x_α, y_β in P
- Let σ be an automorphism of R of order 2. Then A has a superinvolution of the second kind extending σ if and only if there is a rank one R-projective I with a σ-linear automorphism of order 2 (also called σ) a σ-bilinear I form e on P and an element δ in R_0 such that $\sigma(\delta)\delta = 1$ and $\sigma(e(x_\alpha, y_\beta)) = (-1)^{\alpha\beta}\delta e(y_\beta, x_\alpha)$ for all x_α, y_β in P

CONCLUSION

The extended two results proved by Saltman[6] to the supercase and the algebraic K-theory of projective supermodules over (torsion free) supercommutative super-rings would help any researcher to classify further properties about projective supermodules.

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