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Applications of q-Umbral Calculus to Modified Apostol Type q-Bernoulli Polynomials

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Abstract: This article aims to identify the generating function of modified Apostol type q-Bernoulli polynomials. With the aid of this generating function, some properties of modified Apostol type q-Bernoulli polynomials are given. It is shown that aforementioned polynomials are q-Appell. Hence, we make use of these polynomials to have applications on q-Umbral calculus. From those applications, we derive some theorems in order to get Apostol type modified q-Bernoulli polynomials as a linear combination of some known polynomials which we stated in the paper.

Keywords: q-Umbral Calculus, Apostol-Bernoulli Polynomials, Modified Apostol Type q-Bernoulli Polynomials, q-Appell Polynomials, Generating Functions

Introduction

Throughout this paper, we make use of the following standard notations: $\mathbb{N} := \{1,2,3,\cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

We now begin with the fundamental properties of q-calculus. Let q be chosen as a fixed real number between 0 and 1. The q-analogue of any number n is given by:

$$[n]_q = \frac{1-q^n}{1-q}$$

The expression:

$$[n]_q! = [n]_q [n-1]_q \ldots [2]_q [1]_q$$

means the q-factorial of n and also let $n,\ k \in \mathbb{N}_0$, for $k \le n$:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

is called q-binomial coefficient. Note that $[0]_q! := 1$. The q-derivative of $f(x)$ is defined by:

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x)-f(qx)}{(1-q)x} \quad (0<q<1) \tag{1.1}$$

If $q \to 1^-$, it becomes:

$$\lim_{q\to 1^-} D_q f(x) = \frac{df(x)}{dx}$$

representing familiar derivative of a function f, with respect to x. The Jackson definite q-integral of a function f is also defined by:

$$\int_0^a f(x) d_q x = a(1-q)\sum_{j=0}^{\infty} f(q^j a) q^j$$

The q-exponential functions are given by:

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \text{ and } E_q(t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{[n]_q!} \quad (t \in \mathbb{C} \text{ with } |t|<1)$$

with the following equality:

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$$e_{q^{-1}}(t) = E_q(t)$$

These fundamental properties of q-calculus listed above are taken from the book (Kac and Cheung, 2002).

By using an exponential function $e_q(x)$, Kupershmidt (2005) defined the following q-Bernoulli polynomials:

$$\sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{t}{e_q(t)-1} e_q(xt)$$

In the case $x = 0$, $B_{n,q}(0) = B_{n,q}$ means the n-th q-Bernoulli number.

Very recently, Kurt (2016) defined Apostol type q-Bernoulli polynomials of order α by making use of the following generating function:

$$\sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x,y,\lambda) \frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha} e_q(xt) E_q(yt) \quad (1.2)$$

where, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}_0$. In this study, we will study on the following polynomial $B_{n,q}^{(1)}(x,\lambda) := B_{n,q}(x, \lambda)$ which is given by special cases $\alpha = 1$ and $y = 0$ in (1.2):

$$\sum_{n=0}^{\infty} B_{n,q}(x,\lambda) \frac{t^n}{[n]_q!} = \frac{t}{\lambda e_q(t)-1} e_q(xt) \quad (1.3)$$

When $q \to 1$ in (1.3), it reduces to Apostol-Bernoulli polynomials (Choi *et al.*, 2008; Luo and Srivastava, 2006).

We now review briefly the concept of q-umbral calculus. For the properties of q-umbral calculus, we refer the reader to see the references (Araci *et al.*, 2007; Choi *et al.*, 2008; Kac and Cheung, 2002; Kim and Kim, 2014a; Kim *et al.*, 2013; Mahmudov and Keleshteri, 2013; Roman, 1985).

Let $\mathbb{C}$ be a field of characteristic zero and let F be the set of all formal power series in the variable t over $\mathbb{C}$ with:

$$F = \left\{ f \mid f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!},\ (a_k \in \mathbb{C}) \right\}$$

Let $\mathbb{P}$ be the algebra of polynomials in the single variable x over the field complex numbers and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. In the q-Umbral calculus, $\langle L|p(x)\rangle$ means the action of a linear functional L on the polynomial $p(x)$. This operator has a linear property on $\mathbb{P}^*$ given by:

$$\langle L+M \mid p(x)\rangle = \langle L \mid p(x)\rangle + \langle M \mid p(x)\rangle$$

and:

$$\langle cL \mid p(x)\rangle = c\langle L \mid p(x)\rangle$$

for any constant c in $\mathbb{C}$.

The formal power series:

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!} \quad (1.4)$$

defines a linear functional on $\mathbb{P}$ by setting:

$$\langle f(t) \mid x^n\rangle = a_n \quad (x \geq 0) \quad (1.5)$$

Taking $f(t) = t^k$ in Equation 1.4 and 1.5 gives:

$$\langle t^k \mid x^n\rangle = [n]_q!\delta_{n,k}, \quad (n,k \geq 0) \quad (1.6)$$

where:

$$\delta_{n,k} = \begin{cases} 1, & if\ n = k \\ 0, & if\ n \neq k \end{cases}$$

Actually, any linear functional L in $\mathbb{P}^*$ has the form (1.4). That is, since:

$$f_L(t) = \sum_{k=0}^{\infty} \langle L \mid x^k\rangle \frac{t^k}{[k]_q!}$$

we have:

$$\langle f_L(t) \mid x^n\rangle = \langle L \mid x^n\rangle$$

and so as linear functionals $L = f_L(t)$. Moreover, the map $L \to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto F. Henceforth, F will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$ and so an element $f(t)$ of F will be thought of as both a formal power series and a linear functional. From (1.5), we have:

$$\langle e_q(yt) \mid x^n\rangle = y^n$$

and so:

$$\langle e_q(yt) \mid p(x)\rangle = p(y) \quad (p(x) \in \mathbb{P})$$

The order $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series. A series $f(t)$ for which $o(f(t)) = 1$ will be called a delta series (Araci *et al.*, 2007; Choi *et al.*, 2008; Kac and Cheung, 2002; Kim and Kim, 2014a; Kim *et al.*, 2013; Mahmudov and Keleshteri, 2013; Roman, 1985).

If $f_1(t), ..., f_m(t)$ are in F, then:

$$\langle f_1(t)...f_m(t) \mid x_n \rangle = \sum_{i_1+i_2+...+i_m=n} \binom{n}{i_1,...,i_m}_q \langle f_1(t) \mid x^{i_1} \rangle ... \langle f_m(t) \mid x^{i_m} \rangle$$

where:

$$\binom{n}{i_1,...,i_r}_q = \frac{[n]_q!}{[i_1]_q! \cdots [i_r]_q!}$$

We use the notation t^k for the k-th q-derivative operator on $\mathbb{P}$ as follows:

$$t^k x^n = \begin{cases} \frac{[n]_q!}{[n-k]_q!} x^{n-k} & k \le n \\ 0, & k > n \end{cases}$$

If $f(t)$ and $g(t)$ are in F, then:

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle = \langle g(t) \mid f(t)p(x) \rangle$$

for all polynomials $p(x)$. Notice that for all $f(t)$ in F and for all polynomials $p(x)$:

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) \mid x^k \rangle \frac{t^k}{[k]_q!} \quad and \qquad p(x) = \sum_{k=0}^{\infty} \langle t^k \mid p(x) \rangle \frac{x^k}{[k]_q!} \tag{1.7}$$

Using (1.7), we obtain:

$$p^{(k)}(x) = D_q^k p(x) = \sum_{l=k}^{\infty} \frac{\langle t^l \mid p(x) \rangle}{[l]_q!} x^{l-k} \prod_{s=1}^{k} [l-s+1]_q$$

providing:

$$p^{(k)}(0) = \langle t^k \mid p(x) \rangle \text{ and } \langle 1 \mid p^{(k)}(x) \rangle = p^{(k)}(0) \tag{1.8}$$

Thus, from (1.8), we note that:

$$t^k p(x) = p^{(k)}(x) = D_q^k p(x)$$

Let $f(t) \in F$ be a delta series and let $g(t) \in F$ be an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the following property:

$$\langle g(t) f(t)^k \mid s_n(x) \rangle = [n]_q! \delta_{n,k} \quad (x, k \ge 0) \tag{1.9}$$

which is called an orthogonality condition for any q-sheffer sequence, *cf.* (Araci *et al.*, 2007; Choi *et al.*, 2008; Kac and Cheung, 2002; Kim and Kim, 2014a; Kim *et al.*, 2013; Mahmudov and Keleshteri, 2013; Roman, 1985).

The sequence $s_n(x)$ is called the q-Sheffer sequence for the pair of $(g(t), f(t))$, or this $s_n(x)$ is q-Sheffer for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

Let $s_n(x)$ be q-Sheffer for $(g(t), f(t))$. Then for any $h(t)$ in F and for any polynomial $p(x)$, we have:

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid s_k(x) \rangle}{[k]_q!} g(t) f(t)^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k \mid p(x) \rangle}{[k]_q!} s_k(x) \tag{1.10}$$

and the sequence $s_n(x)$ is q-Sheffer for $(g(t), f(t))$ if and only if:

$$\frac{1}{g(\bar{f}(t))} e_q(x\bar{f}(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t_n}{[n]_q!} \tag{1.11}$$

for all x in $\mathbb{C}$, where $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

An important property for the q-Sheffer sequence $s_n(x)$ having $(g(t), t)$ is the q-Appell sequence. It is also called q-Appell for $g(t)$ with the following consequence:

$$s_n(x) = \frac{1}{g(t)} x^n \Leftrightarrow t s_n(x) = [n]_q s_{n-1}(x) \tag{1.12}$$

Further important property for q-Sheffer sequence $s_n(x)$ is as follows:

$$s_n(x) is\, q - Alppel\, for\, g(t) \Leftrightarrow \frac{1}{g(t)} e_q(xt) = \sum_{k=0}^{\infty} s_n(x) \frac{t^n}{[n]_q!} (x \in \mathbb{C})$$

For having information about the properties of q-umbral theory (Araci *et al.*, 2007; Choi *et al.*, 2008; Kac and Cheung, 2002; Kim and Kim, 2014a; Kim *et al.*, 2013; Mahmudov and Keleshteri, 2013; Roman, 1985) and cited references therein.

Recently several authors have studied q-Bernoulli polynomials, q-Euler polynomials and various generalizations of these polynomials (Araci *et al*., 2007; Choi *et al*., 2008; Kac and Cheung, 2002; Kim and Kim, 2014a; 2014b; 2015; Kim *et al*., 2013; Kurt, 2016; Kurt and Simsek, 2013; Kupershmidt, 2005; Luo and Srivastava, 2006; Mahmudov, 2013; Mahmudov and Keleshteri, 2013; Roman, 1985; Srivastava, 2011). In the next section, we investigate modified Apostol type q-Bernoulli numbers and polynomials and we apply these numbers and polynomials to q-umbral theory which is the systematic study of q-umbral algebra. Actually, we are motivated to write this paper from Kim's systematic works on q-umbral theory (Kim and Kim, 2014a; 2014b; 2015; Kim *et al*., 2013).

Modified Apostol Type *q*-Bernoulli Numbers and Polynomials

Recall from (1.3) that:

$$\sum_{n=0}^{\infty} B_{n,q}(x,\lambda)\frac{t^n}{[n]_q!}=\frac{t}{\lambda e_q(t)-1}e_q(xt)\ \ (\lambda\neq 1) \tag{2.1}$$

Taking $t\to 0$ on the above gives $B_{0,q}(x,\lambda)=0$. This shows that the generating function of these polynomials is not invertible. Therefore, we need to modify slightly Equation (2.1) as follows:

$$F_q^*(x,t)=\sum_{n=0}^{\infty} B^*_{n,q}(x,\lambda)\frac{t^n}{[n]_q!}=\frac{t}{\lambda e_q(t)-1}e_q(xt)$$

representing:

$$\frac{B_{n+1,q}(x,\lambda)}{[n+1]_q}=B^*_{n,q}(x,\lambda) \tag{2.2}$$

Here we called $B^*_{n,q}(x,\lambda)$ modified Apostol type q-Bernoulli polynomials. Now:

$$\lim_{t\to 0}F_q^*(x,t)=B^*_{0,q}(x,\lambda)=\frac{1}{\lambda-1}\neq 0\ \ (\lambda\neq 1)$$

This modification yields to being invertible for generating function of modified Apostol type q-Bernoulli polynomials. As a traditional for some special polynomials to be a number, in the case when $x=0$, $B^*_{n,q}(0,\lambda)=B^*_{n,q}(\lambda)$ is called the modified Apostol type n-th q-Bernoulli number. Now we list some properties of modified Apostol type q-Bernoulli polynomials as follows.

From (2.2), we obtain:

$$B^*_{n,q}(x,\lambda)=\sum_{k=0}^{n}\binom{n}{k}_q B^*_{k,q}(\lambda)x^{n-k}=\sum_{k=0}^{n}\binom{n}{k}_q B^*_{n-k,q}(\lambda)x^{k} \tag{2.3}$$

By (2.2), the modified Apostol type q-Bernoulli numbers can be found by means of the following recurrence relation:

$$B^*_{0,q}(x,\lambda)=\frac{1}{\lambda-1}\ \textit{and}\ \lambda B^*_{n,q}(1,\lambda)-B^*_{n,q}(\lambda)=\delta_{0,n} \tag{2.4}$$

A few numbers are listed below:

$$B^*_{0,q}(\lambda)=\frac{1}{\lambda-1},B^*_{1,q}(\lambda)=\frac{-\lambda}{(\lambda-1)^2},B^*_{2,q}(\lambda)=\frac{\lambda(1+\lambda q)}{(\lambda-1)^3}$$
$$B^*_{3,q}(\lambda)=\frac{-\lambda(1+2\lambda q+2\lambda q^2+\lambda^2q^3)}{(\lambda-1)^4}$$

From (1.11) and (1.12), we have:

$$B^*_{n,q}(x,\lambda)\sim(\lambda e_q(t)-1,t) \tag{2.5}$$

and:

$$tB^*_{n,q}(x,\lambda)=[n]_q B^*_{n-1,q}(x,\lambda)=B_{n,q}(x,\lambda) \tag{2.6}$$

It follows from (2.6) that $B^*_{n,q}(x,\lambda)$ is q-Appell for $\lambda e_q(t)$-1.

We now have the following theorem.

Theorem 1

Let $p(x)\in\mathbb{P}$. We have:

$$\left\langle \frac{\lambda e_q(t)-1}{t}\,|\,p(x)\right\rangle=\lambda\int_0^1 p(u)d_qu$$

Proof

From Equation (2.5) and (2.6), we write:

$$B^*_{n,q}(x,\lambda)=\frac{1}{\lambda e_q(t)-1}x^n\ \ (n\geq 0)$$

By (1.1) and (1.6), we obtain the following calculations:

$$\begin{aligned}\left\langle \frac{\lambda e_q(t)-1}{t}\,|\,x^n\right\rangle&=\frac{1}{[n+1]_q}\left\langle \frac{\lambda e_q(t)-1}{t}\,|\,tx^{n+1}\right\rangle\\&=\frac{1}{[n+1]_q}\left\langle \lambda e_q(t)-1\,|\,x^{n+1}\right\rangle\\&=\frac{\lambda}{[n+1]_q}=\lambda\int_0^1 x^n d_qx\end{aligned} \tag{2.7}$$

Thus, from (2.7), we arrive at:

$$\left\langle \frac{\lambda e_q(t)-1}{t} \mid p(x) \right\rangle = \lambda \int_0^1 p(u)\, d_q u \quad (p(x) \in \mathbb{P})$$

which is desired result.

Example 1

If we take $p(x) = B^*_{n,q}(x,\lambda)$ in Theorem 1, on the one hand, we derive:

$$\lambda \int_0^1 B^*_{n,q}(x,\lambda)\, d_q x = \left\langle \frac{\lambda e_q(t)-1}{t} \mid B^*_{n,q}(x,\lambda) \right\rangle$$

$$= \left\langle 1 \mid \frac{\lambda e_q(t)-1}{t} \frac{t B^*_{n+1,q}(x,\lambda)}{[n+1]_q} \right\rangle$$

$$= \frac{1}{[n+1]_q} \left\langle t^0 \mid x^{n+1} \right\rangle = [n]_q!\, \delta_{n+1,0}$$

On the other hand:

$$\lambda [n+1]_q \int_0^1 B^*_{n,q}(x,\lambda)\, d_q x = \lambda [n+1]_q \int_0^1 \sum_{k=0}^{n} \binom{n}{k}_q B^*_{n-k,q}(\lambda) x^k d_q x$$

$$= \lambda [n+1]_q \sum_{k=0}^{n} \binom{n}{k}_q B_{n-k,q}(\lambda) \int_0^1 x^k d_q x$$

$$= \lambda \sum_{k=0}^{n} \binom{n+1}{k+1}_q B^*_{n-k,q}(\lambda)$$

Thus we have the following interesting property for modified Apostol type q-Bernoulli numbers derived from Theorem 1 for $n \geq 0$:

$$\sum_{k=0}^{n} \binom{n+1}{k+1}_q B^*_{n-k,q}(\lambda) = 0$$

which can be also generated by Equation (2.3) and (2.4).

The following is an immediate result emerging from (1.10) and (2.5) that:

$$p(x) = \sum_{k=0}^{\infty} \frac{[n+1]_q}{[k]_q!} \left\langle \frac{\lambda e_q(t)-1}{t} t^k \mid p(x) \right\rangle B^*_{k,q}(x,\lambda)$$

$$= \sum_{k=0}^{\infty} \frac{[n+1]_q}{[k]_q!} \left\langle \frac{\lambda e_q(t)-1}{t} \mid t^k p(x) \right\rangle B^*_{k,q}(x,\lambda)$$

$$= \lambda \sum_{k=0}^{\infty} \frac{[n+1]_q}{[k]_q!} B^*_{k,q}(x,\lambda) \int_0^1 t^k p(x)\, d_q x$$

By choosing suitable polynomials $p(x)$, one can derive some interesting results. So we omit to give examples and so we now take care of a fundamental property in q-umbral theory which is stated below by Theorem 2.

Theorem 2

Let n be nonnegative integer. Then we have:

$$\left\langle \frac{e_q(t)-1}{t} \mid B^*_{n,q}(x,\lambda) \right\rangle = \int_0^1 B^*_{n,q}(u,\lambda)\, d_q u$$

Proof

From (2.3), we first obtain:

$$\int_x^{x+y} B^*_{n,q}(u,\lambda) d_q u = \sum_{k=0}^{n} \binom{n}{k}_q B^*_{n-k,q}(\lambda) \frac{1}{[k+1]_q} \left\{ (x+y)^{k+1} - x^{k+1} \right\}$$

$$= \frac{1}{[n+1]_q} \sum_{k=0}^{n} \binom{n+1}{k+1}_q B^*_{n-k,q}(\lambda) \left\{ (x+y)^{k+1} - x^{k+1} \right\}$$

$$= \frac{1}{[n+1]_q} \left(B^*_{n-1,q}(x+y,\lambda) - B^*_{n+1,q}(x,\lambda) \right)$$

Thus, by applying (2.8), we get:

$$\left\langle \frac{e_q(t)-1}{t} \mid B^*_{n,q}(x,y) \right\rangle = \frac{1}{[n+1]_q} \left\langle \frac{e_q(t)-1}{1} \mid t B^*_{n+1,q}(x,\lambda) \right\rangle$$

$$= \frac{1}{[n+1]_q} \left\{ B^*_{n+1,q}(1,\lambda) - B^*_{n+1,q}(\lambda) \right\} \tag{2.9}$$

$$= \int_0^1 B^*_{n,q}(u,\lambda)\, d_q u$$

Comparing Equation (2.8) with Equation (2.9), we complete the proof of this theorem.

The following theorem is useful to derive any polynomial as a linear combination of modified Apostol type q-Bernoulli polynomials.

Theorem 3

For $q(x) \in P_n$, let:

$$q(x) = \sum_{k=0}^{n} b_{k,q} B^*_{k,q}(x,\lambda)$$

Then:

$$b_{k,q} = \frac{1}{[k]_q!} \left\{ \lambda q^{(k)}(1) - q^{(k)}(0) \right\}$$

Proof

It follows from (1.9) that:

$$\left\langle \left(\lambda e_q(t)-1\right) t^k \mid B^*_{n,q}(x,\lambda) \right\rangle = [n]_q!\, \delta_{n,k} \quad (n,k \geq 0) \tag{2.10}$$

We now consider the following sets of polynomials of degree less than or equal to n:

$$\mathbb{P}_n = \{q(x) \in \mathbb{C}[x] \mid \deg q(x) \le n\}$$

For $q(x) \in \mathbb{P}_n$, we further consider that:

$$q(x) = \sum_{k=0}^{n} b_{k,q} B^*_{k,q}(x,\lambda) \tag{2.11}$$

Combining (2.10) with (2.11), it becomes:

$$\begin{aligned} \left\langle \left(\lambda e_q(t)-1\right)t^k \mid q(x)\right\rangle &= \sum_{l=0}^{n} b_{l,q} \left\langle \left(\lambda e_q(t)-1\right)t^k \mid B^*_{l,q}(x,\lambda)\right\rangle \\ &= \sum_{l=0}^{n} b_{l,q}[l]_q!\delta_{l,k} = [k]_q! b_{k,q} \end{aligned} \tag{2.12}$$

Thus, from (2.12), we have:

$$b_{k,q} = \frac{1}{[k]_q!}\left\langle \left(\lambda e_q(t)-1\right)t^k \mid q(x)\right\rangle = \frac{1}{[k]_q!}\left\{\lambda q^{(k)}(1) - q^{(k)}(0)\right\}$$

where, $q^{(k)}(x) = D_q^k q(x)$. Thus the proof is completed.

When we choose $q(x) = E_{n,q}(x)$, we have the following corollary which is given by its proof.

Corollary 1

Let $n \ge 2$. Then:

$$\begin{aligned} E_{n,q}(x) &= (\lambda q - 1) B^*_{n,q}(x,\lambda) + [n]_q \left(\frac{\lambda+1}{2}\right) B^*_{n-1,q}(x,\lambda) \\ &-(\lambda+1)\sum_{k=0}^{n-2} \binom{n}{k}_q E_{n-k,q} B^*_{k,q}(x,\lambda) \end{aligned}$$

Proof

Recall that the q-Euler polynomials $E_{n,q}(x)$ are defined by (Mahmudov, 2013; Srivastava, 2011):

$$\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(t)+1} e_q(xt)$$

which in turn yields to:

$$E_{n,q}(x) \sim \left(\frac{e_q(t)+1}{[2]_q}, t\right) \quad (n \ge 0)$$

and:

$$tE_{n,q}(x) = [n]_q E_{n-1,q}(x)$$

Set:

$$q(x) = E_{n,q}(x) \in \mathbb{P}_n$$

Then it becomes:

$$E_{n,q}(x) = \sum_{k=0}^{n} b_{k,q} B^*_{k,q}(x,\lambda) \tag{2.13}$$

Let us now evaluate the coefficients $b_{k,q}$ as follows:

$$\begin{aligned} b_{k,q} &= \frac{1}{[k]_q!}\left\langle \left(\lambda e_q(t)-1\right)t^k \mid E_{n,q}(x)\right\rangle \\ &= \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!}\left\langle \lambda e_q(t)-1 \mid E_{n-k,q}(x)\right\rangle \\ &= \binom{n}{k}_q \left\langle \lambda e_q(t)-1 \mid E_{n-k,q}(x)\right\rangle \\ &= \binom{n}{k}_q \left(\lambda E_{n-k,q}(1) - E_{n-k,q}\right) \end{aligned}$$

where, $E_{n,q} := E_{n,q}(0)$ are called q-Euler numbers satisfying the following property:

$$E_{n,q}(1) + E_{n,q} = [2]_q \delta_{0,n} \tag{2.14}$$

with the conditions $E_{0,q} = 1$ and $E_{1,q} = -\frac{1}{2}$. By (2.13) and (2.14), we have:

$$\begin{aligned} E_{n,q}(x) &= b_{n,q}B^*_{n,q}(x,\lambda) + b_{n-1,q}B^*_{n-1,q}(x,\lambda) + \sum_{k=0}^{n-2} b_{k,q}B^*_{k,q}(x,\lambda) \\ &= (\lambda q-1)B^*_{n,q}(x,\lambda) + [n]_q\left(\frac{\lambda+1}{2}\right)B^*_{n-1,q}(x,\lambda) \\ &-(\lambda+1)\sum_{k=0}^{n-2}\binom{n}{k}_q E_{n-k,q}B^*_{k,q}(x,\lambda). \end{aligned}$$

Recall from (1.2) that Apostol type q-Bernoulli polynomials of order r are given by the following generating function, for $y = 0$ (Kurt, 2016):

$$\sum_{n=0}^{\infty} B^{(r)}_{n,q}(x,\lambda)\frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t)-1}\right)^r e_q(xt)$$

where, $t \in \mathbb{C}$ and $r \in \mathbb{N}$. If t approaches to 0 on the above, it yields to $B^{(\alpha)}_{0,q}(x,\lambda) = 0$, which means that the generating function of $B^{(\alpha)}_{n,q}(x,\lambda)$ is not invertible. So, we need to modify slightly Equation (2.1), as follows:

$$\overline{F}_q^{(r)}(x,t)=\sum_{n=0}^{\infty}\overline{B}_{n,q}^{(r)}(x,\lambda)\frac{t^n}{[n]_q!}=\left(\frac{1}{\lambda e_q(t)-1}\right)^r e_q(xt) \quad (2.15)$$

which implies an invertible since:

$$\lim_{t\to 0}\overline{F}_q^{(r)}(x,t)=\overline{B}_{n,q}^{(r)}(x,\lambda)=\left(\frac{1}{\lambda-1}\right)^r\neq 0 \ \ (\lambda\neq 1)$$

Therefore, we called $\overline{B}_{n,q}^{(r)}(x,\lambda)$ as modified Apostol type q-Bernoulli polynomials of higher order. In the case $x = 0, \overline{B}_{n,q}^{(r)}(0,\lambda) := \overline{B}_{n,q}^{(r)}(\lambda)$ may be called the modified Apostol type q-Bernoulli numbers.

Let:

$$g^r(t,\lambda)=\left(\lambda e_q(t)-1\right)^r$$

It is clear that $g^r(t, \lambda)$ is an invertible series. It follows from (2.15) that $\overline{B}_{n,q}^{(r)}(x,\lambda)$ is q-Appell for $(\lambda e_q(t) -1)^r$. So, by (1.12), we have:

$$\overline{B}_{n,q}^{(r)}(x,\lambda)=\frac{1}{g^r(t,\lambda)}x^n$$

and:

$$t\overline{B}_{n,q}^{(r)}(x,\lambda)=[n]_q\overline{B}_{n-1,q}^{(r)}(x,\lambda)$$

Thus, we have:

$$\overline{B}_{n,q}^{(r)}(x,\lambda)\sim\left(\left(\lambda e_q(t)-1\right)^r,t\right)$$

By (1.5) and (2.15), we get:

$$\begin{aligned}&\left\langle\frac{1^r}{\left(\lambda e_q(t)-1\right)^r}e_q(yt)\,|\,x^n\right\rangle\\&=\overline{B}_{n,q}^{(r)}(y,\lambda)=\sum_{l=0}^{n}\binom{n}{l}_q\overline{B}_{n-l,q}^{(r)}(\lambda)y^l\end{aligned} \quad (2.16)$$

Here we find that:

$$\begin{aligned}&\left\langle\left(\frac{1}{\lambda e_q(t)-1}\right)^r|\,x^n\right\rangle=\left\langle\frac{1}{\lambda e_q(t)-1}\times\cdots\times\frac{1}{\lambda e_q(t)-1}\,|\,x^n\right\rangle\\&=\sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\cdots,i_r}_q B_{i_1,q}^{*}(\lambda)\cdots B_{i_r,q}^{*}(\lambda)\end{aligned} \quad (2.17)$$

By using (2.16), we have:

$$\left\langle\left(\frac{1}{\lambda e_q(t)-1}\right)^r|\,x^n\right\rangle=\overline{B}_{n,q}^{(r)}(\lambda) \quad (2.18)$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 4

Let n be nonnegative integer. Then we have:

$$\overline{B}_{n,q}^{(r)}(\lambda)=\sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\cdots,i_r}_q\prod_{j=1}^{r}B_{i_j,q}^{*}(\lambda)$$

Set:

$$q(x)=\overline{B}_{n,q}^{(r)}(x,\lambda)\in\mathbb{P}_n$$

Then, by Theorem 3, we write:

$$\overline{B}_{n,q}^{(r)}(x,\lambda)=\sum_{k=0}^{n}b_{k,q}B_{k,q}^{*}(x,\lambda) \quad (2.19)$$

where the coefficient $b_{k,q}$ is given by:

$$\begin{aligned}b_{k,q}&=\frac{1}{[k]_q!}\left\langle\left(\lambda e_q(t)-1\right)t^k\,|\,q(x)\right\rangle\\&=\binom{n}{k}_q\left\langle\left(\lambda e_q(t)-1\right)|\,\overline{B}_{n-k,q}^{(r)}(x,\lambda)\right\rangle\\&=\binom{n}{k}_q\left(\lambda\overline{B}_{n-k,q}^{(r)}(1-\lambda)-\overline{B}_{n-k,q}^{(r)}(\lambda)\right)\end{aligned} \quad (2.20)$$

From the Equation (2.15), we have:

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\lambda\overline{B}_{n,q}^{(r)}(1,\lambda)-\tilde{B}_{n,q}^{(r)}(\lambda)\right)\frac{t^n}{[n]_q!}=\left(\frac{1}{\lambda e_q(t)-1}\right)^r\left(\lambda e_q(t)-1\right)\\&=\left(\frac{1}{\lambda e_q(t)-1}\right)^{r-1}\\&=\sum_{n=0}^{\infty}\tilde{B}_{n,q}^{(r-1)}(\lambda)\frac{t^n}{[n]_q!}\end{aligned}$$

By comparing the coefficients $\frac{t^n}{[n]_q!}$ in the above equation, we get:

$$\lambda\overline{B}_{n,q}^{(r)}(1,\lambda)-\overline{B}_{n,q}^{(r)}(\lambda)=\overline{B}_{n,q}^{(r-1)}(\lambda) \quad (2.21)$$

From the Equation (2.19) to (2.21), we get the following theorem.

Theorem 5

Let $n \in \mathbb{N}_0$ and $r \in \mathbb{N}_0$. Then:

$$\overline{B}_{n,q}^{(r)}(x,\lambda) = \sum_{k=0}^{n} \binom{n}{k}_q \overline{B}_{n-k,q}^{(r-1)}(\lambda) B_{k,q}^{*}(x,\lambda)$$

Let us assume that:

$$q(x)\sum_{k=0}^{n} b_{k,q}^{r} \overline{B}_{k,q}^{(r)}(x,\lambda) \in \mathbb{P}_n \quad (2.22)$$

We use a similar method in order to find the coefficient $b_{k,q}^{r}$ as same as Theorem 3. So we omit the details and give the following equality:

$$b_{k,q}^{r} = \frac{1}{[k]_q!}\sum_{l=0}^{r}\binom{r}{l}_q \lambda^l (-1)^{r-l} \sum_{m\geq 0}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\cdots,i_l}_q \frac{1}{[m]_q!} q^{(m+k)}(0)$$

By (2.22) and coefficient $b_{k,q}^{r}$, we state the following theorem.

Theorem 6

For $n \in \mathbb{N}_0$, let:

$$q(x) = \sum_{k=0}^{n} b_{k,q}^{r} \overline{B}_{k,q}^{(r)}(x,\lambda) \in \mathbb{P}_n$$

Then:

$$b_{k,q}^{r} = \frac{1}{[k]_q!}\left\langle \left(\lambda e_q(t)-1\right) t^k \mid q(x)\right\rangle$$

$$= \frac{1}{[k]_q!}\sum_{m\geq 0}\sum_{l=0}\binom{r}{l}_q \lambda^l (-1)^{r-l} \sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\cdots,i_l}_q \frac{1}{[m]_q!} q^{(m+k)}(0)$$

where, $q^{(k)}(x) = D_q^k q(x)$.

Let us consider $q(x) = B_{n,q}^{*}(x,\lambda) \in \mathbb{P}_n$. Then, by Theorem 6, we have:

$$B_{n,q}^{*}(x,\lambda) = \sum_{k=0}^{n} b_{k,q}^{r} \overline{B}_{k,q}^{(r)}(x,\lambda) \quad (2.23)$$

From Theorem 6 and (2.23), we acquire the following theorem.

Theorem 7

For n, $r \in \mathbb{N}_0$, the following equality holds true:

$$B_{n,q}^{*}(x,\lambda) = \sum_{k=0}^{n}\left(\sum_{m=0}^{n-k}\sum_{l=0}^{r}\sum_{i_1+\cdots+i_l=m}(-1)^{r-l}\lambda^l\binom{r}{l}_q\binom{m}{i_1,\cdots,i_l}_q \times \binom{m+k}{m}_q \binom{n}{m+k}_q B_{n-m-k,q}^{*}(\lambda)\right)\overline{B}_{k,q}^{(r)}(x,\lambda)$$

Conclusion

In the paper, we have derived some new and interesting identities arising from q-umbral theory.

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Author's Contributions

All authors equally contributed to this paper.

Competing Interests

The authors have no competing interests.

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