Journal of Mathematics and Statistics

Original Research Paper

Mathematical Analysis of *M/M/C* Vacation Queueing Model with a Waiting Server and Impatient Customers

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Article history
Received: 17-05-2021
Revised: 02-08-2021
Accepted: 20-08-2021

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Abstract: In this study, the transient analysis of the *M/M/C* queueing system has been made under the provision of servers' single vacation and loss of impatient customers. Customers arrive in the system in the Poisson process and are served by multiple servers in an exponential distribution process. Customers are served in the chronological order of their arrival. The main purpose of this investigation is to derive (i) the probability distribution functions, (ii) the formulas for the expected number of the customers in the system as well as in queue in the explicit form, (iii) the expected sojourn time and the expected time spent in waiting in the queue. Moreover, the sensitiveness of performance measures due to the small change of vacation rate γ, impatient rate ξ, and server's waiting rate η has also been shown graphically. To show the applicability of the model under study, ample numerical results have been illustrated. The error computations have also been cited during the vacation period and busy period. Queueing model understudy may have its applications in multichannel telecommunications, security systems in the airport, train stations, and the manufacturing system.

Keywords: Transient, Queue, Vacation, Poisson Distribution, Sojourn Time

Introduction

In real-life, many queueing situations arise in which there may be a tendency for customers to be discouraged by a long queue. As a result, the customers either decide not to join the queue (balking) or depart after joining the queue without getting service due to impatience (reneging). Balking and reneging are not only common phenomena in queues rising in daily activities, but also in various machine models it has been prevailing. Many practical queueing systems especially those with balking and reneging have been widely applied to many real-life problems, such as the hospital emergency rooms handling critical patients and the inventory systems with storage of perishable goods. Several researchers in the past have studied queueing models with impatient customers, where the cause of impatience was considered to be either a heavy load of the server leading to a long wait that is experienced by the customers already in a queue or due to anticipation of impatience by a customer upon arrival so it is worthwhile to report some of the works done on the line. Perel and Yechiali (2010) analyzed customers' impatience that arises as a result of a slowdown in the rate of the server in a 2-phases (fast and slow) Markovian tandem environment. Dequan Yue *et al*. (2006) analyzed *M/M/c/N* queueing system with balking, reneging, and synchronous vacations of partial servers. Astakhov and Pichkurova (2021) dealt with the deformation characteristics of reinforcement coupling with concrete for railway transport structures and they presented the results of experimental studies of cable armature coupling in pres-tress cylindrical samples. Shin and Choo (2009) analyzed numerically an *M/M/s* queue with balking, reneging, and retrials by using an algorithm based on the generalized truncation method. A vacation queueing system is one in which a server may become unavailable for a random period from a primary service center. The time away from the primary service center is termed vacation. In some cases, the vacation can be occurred due to a server breakdown, which means that the system must be repaired and brought back to service. It can also be a deliberate action taken to utilize the server in a secondary service center when there are no customers present at the primary service center. Several authors have been attracted to the contributions to vacation queueing systems due to their applications in modern complex technological advancements. Wu and Ke (2013) studied an infinite buffer *M/M/c* queueing system under a multiple-threshold synchronous vacation policy with partial servers taking a synchronous single vacation when the number of customers in the system is less than a pre-fixed threshold and obtained a closed-form expression of the rate matrix by using the matrix-analytical method. Manoharan and Mojid (2017) contributed to a stationary analysis of a multi-server queue with multiple working vacations. Shanmugasundaram and Venkatesh (2016) investigated a multi-server queue with a single vacation policy in a fuzzy environment and used the approximation technique in the form of an algorithm to define membership functions of the performance measures such as expected waiting time in the queue and impatient customers. Raj and Chandrasekhar (2015) studied N-policy multiple vacation queueing system with breakdown and repair. Wu and Ke (2014) considered a multi-server machine repair queueing problem with M operating machines and S standbys in which R repairmen were responsible for supervising these machines and operating (V, R) vacation policy. Recently Bouchentouf *et al.* (2020) developed the M/M/1/N/DWN Markovian queueing model with the assumptions of Bernoulli scheduled vacation interruption and working vacation after the busy period under balking and reneging and determined the optimal service rates and vacation rates. Panta *et al*. (2021) developed a multi-server Markovian queueing model under the fuzzy environment with the provision of reneging of customers and they derived some performance measures in explicit form.

Sindhu *et al*. (2021) proposed an exponentiated transformation of Gumbel Type-II (ETGTII) distribution for modeling COVID-19 and model parameters were estimated by utilizing the maximum likelihood method and Bayesian paradigm. Kumar and Madheswari (2002) obtained a transient solution for the system size in the *M/M/*2 queue with the possibility of catastrophes at the service stations using the probability generating function and Modified Bessel function. Kumar *et al*. (2014) discussed the transient probabilities for a single server queueing system subject to catastrophic failures and impatience of customers. Ammar (2017) discussed the transient solution of an *M/M/*1 vacation queue with a waiting server and impatient customer and he derived closed-form explicit expressions analytically for the system size probabilities, mean and variance in terms of Modified Bessel functions employing Laplace transform, continued fractions, generating functions. In the same way, Ammar (2015) carried out the transient analysis of an M/M/1 queue with impatient customers and multiple vacations where customers' impatience was due to an absence of a server upon arrival. Liu *et al*. (2021) discussed the cooperative hypercube queueing model for emergency service systems and verified the accuracy of the model by applying Arena simulation software. Sah and Ghimire (2015) analyzed the transient $M/E_k/1$ queueing model and obtained some of the performance measures of the model by using the probability generating function. Al-Seedy *et al*. (2009) presented the transient solution of an M/M/c queue with balking and reneging using generating function technique in terms of modified Bessel functions. Jindal *et al*. (2016) analyzed the time-dependent infinite waiting space multi-server feedback queueing problem with reneging. They demonstrated the technique of obtaining stability conditions and probability distributions of the states of a two-dimensional non-homogeneous queueing system by using the underlying Quasi-Birth-Death (QBD) structure and the matrix-geometric approach. A detailed study of the matrix-geometric approach can be made in Neuts (1981). Shoukry *et al*. (2018) used the matrix-geometric method to derive the stationary distribution of the *M/M/*1 queueing model with breakdown using the transition structure of its Markov chain. Working vacation policy is the rule under which the server resumes its work at the normal service rate after the end of the vacation, only if the customers are waiting in the queue. Liu and Hlynka (2018) formulated a queueing model as a quasi-birth-death process and made the use of the partial probability generating functions for the distribution of queue sizes when the server is in a working vacation period and regular busy period. They also obtained many performance measures such as the mean queue length and mean waiting time by using matrix-analytic methods for the solution of an arising system of differential equations. Vijayashree and Janani (2015) considered an *M/M/c* queueing model subject to multiple exponential working vacations and they obtained the time-dependent probabilities of the number of customers in the system in the Laplace domain using the matrix-geometric method. Shah *et al*. (2009) studied the performance evaluation of multistage service systems using the matrix-geometric method. Abate and Whitt (1992) suggested the Fourier-series method for computation of cumulative distribution functions and probability mass functions numerically by inverting the characteristics functions, the Laplace transform, and the generating functions. They presented the numerical technique for inversion of Laplace transform and identified the Poisson summation formula for discretization error associated with the Trapezoidal rule. Recently, Haralambie and Mandjes (2019) introduced a class of Markov processes modeling the time evolution of the network configuration of any open, work-conservative Multi-Class Queueing Network (McQN) having exponential service times and Poisson input.

The scope and purpose of this study are to derive (i) the probability distribution functions, (ii) the formulas for the expected number of the customers in the system as well as in queue in the explicit form, (iii) the expected sojourn time and the expected time spent in waiting in the queue. Moreover, the sensitiveness of performance measures due to the small change of vacation rate γ, impatient rate ξ, and server's waiting rate η has also been shown graphically. To show the applicability of the model under study, ample numerical results have been illustrated. Our investigation is novel in the way that we have triggered the problem in the case of multi-servers with a time-dependent framework. The techniques we have used have rarely been applied.

Model Description

Mathematical Model

The *M/M/c* vacation queueing model with a waiting server and impatient customers can be modeled by a two dimensional continuous-time Markov process $\{X(t), n(t), t \geq 0\}$, where $n(t)$ is the number of units in the system at time t and $X(t)$ is the system state at time t. If $X(t) = 0$, the server is on vacation, whilst if $X(t) = 1$, the server is working and serving customers. Let $P_{i,j}(t) = P[X(t) = i, n(t) = j]$ denote the system state in the transient probabilities.

Basic Assumptions

For our model we have made the following assumptions:

(a) Customers arriving according to a Poisson process with rate λ, the server has an independently and identically distributed exponential service time distribution with the rate μ and mean service discipline is FCFS and, there is infinite room for customers to wait
(b) When the busy period is ended the server waits a random duration of time before beginning a vacation. This waiting duration follows the exponentially distributed with the density function is given by $w(t) = \eta e^{-\eta t} t \geq 0, \eta \geq 0$, where η is the waiting rate of a server
(c) It is assumed that the interval of vacation has an exponential distribution with the density function is given by $v(t) = \gamma e^{-\gamma t}\, t \geq 0, \gamma \geq 0$, where γ is the vacation rate of a server
(d) When the server is on a vacation, each customer sets up an impatience timer independently of the other customers in the system, which is assumed to be exponentially distributed with the density function is given by $s(t) = \xi e^{-\xi t}\, t \geq 0,\ \xi \geq 0$, where ξ is the impatience rate of a customer
(e) If the impatience time expires while the server is on a vacation, the customer abandons the queue, never to return

The system of differential equations for the model obtained from the above transition diagram is: $dP_0,0(t)$

$$\frac{dP0,0(t)}{dt} = -(\lambda+\gamma)P0,0(t) + \eta P1,0(t) + \xi P0,1(t) \tag{1}$$

$$\frac{dP0,n(t)}{dt} = \lambda P0,_{n-1}(t) - (\lambda+\xi+\gamma) P0,n(t) + \xi P0,_{n+1}(t); n \geq 1 \tag{2}$$

$$\frac{dP1,n(t)}{dt} = \gamma P0,0(t) - (\lambda+\eta)P_{1,0}(t) + \mu P_{1,1}(t) \tag{3}$$

$$\frac{dP1,n(t)}{dt} = \lambda P_{1,n-1}(t) + \gamma P_{0,n}(t) - (\lambda + n\mu) P1,n(t) + (n+1)\mu P1,n+1(t)\ for \leq n \leq c-1 \tag{4}$$

$$\frac{dP1,n(t)}{dt} = \lambda P_{1,n-1}(t) + \gamma P_{0,n}(t) - (\lambda + c\mu) P1,n(t) + c\mu P_{1,n+1}(t)\ for\, n \geq c. \tag{5}$$

Let $P_n(t) = [P_{0,n}(t), P_{1,n}(t)];\ n = 0,1,2,\cdots$.

The above system of equations can be represented in the matrix form as:

$$\frac{d\,\overrightarrow{p(t)}}{dt} = \vec{P}(t)\mathbb{Q} \tag{6}$$

where, $\vec{P}(t) = \left[\vec{P}_0(t)\vec{P}_1(t)\vec{P}_2(t)\ldots\right]$ and the infinitesimal generator Q is given by:

$$\begin{bmatrix} A_0 C & & & & & & \\ B_1\ A_1\ C & & & & & & \\ & B_2\ A_2\ C & & & & & \\ & & B_3\ A_3\ C & & & & \\ & & . \ . \ . & & & & \\ & & . \ . \ . & & & & \\ & & . \ . \ . & & & & \\ & & & B_{c-1} A_{c-1} C & & & \\ & & & & B\ A\ C & & \\ & & & & . \ . \ . & & \\ & & & & . \ . \ . & & \\ & & & & . \ . \ . & & \end{bmatrix}$$

Where:

$$A_0 = \begin{bmatrix} -(\lambda+\gamma) & \gamma \\ \eta & (\lambda+\eta) \end{bmatrix}, C = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_n = \begin{bmatrix} -(\lambda+\xi+\gamma) & \gamma \\ 0 & (\lambda+n\eta) \end{bmatrix}; n = 1,2,3,\ldots,c-1,$$

$$B_n = \begin{bmatrix} \xi & 0 \\ 0 & n\mu \end{bmatrix}; n = 1,2,3,\ldots,c-1,$$

$$A = \begin{bmatrix} -(\lambda+\xi+\gamma) & \gamma \\ 0 & (\lambda+c\mu) \end{bmatrix}, B = \begin{bmatrix} \xi & 0 \\ 0 & c\mu \end{bmatrix}$$

Transient Analysis

Let, $\hat{\vec{P}}_n(s)$ denote the Laplace transform of $\vec{P}_n(t)$ for $n = 0,1,2,\cdots$. Laplace transform of Eq. (6)

$$s\overleftarrow{P}(s)-\vec{P}(0)=\overleftarrow{P}(s)\mathbb{Q}\Rightarrow \overleftarrow{P}(s)\left[\mathbb{Q}-sI\right]=-\vec{P}(0)$$

$$\Rightarrow\left[\overleftarrow{P_0}(s)[\mathbb{Q}-sI]\right]=-\left[\overrightarrow{P_0}(0)\overrightarrow{P_1}(0)\overrightarrow{P_2}(0)\ldots\right]$$

where, $\mathbb{Q}-sI$ is:

$$\begin{bmatrix} A_0-sI & C & & & & & \\ B_1 & A_1-sI & C & & & & \\ & B_2 & A_2-sI & C & & & \\ & & B_3 & A_3-sI & C & & \\ & & & \ddots & \ddots & \ddots & \\ & & & B_{c-1} & A_{c-1}-sI & C & \\ & & & & B & A-sI & C \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\Rightarrow \overrightarrow{P_0}(s)(A0-sI)+\overrightarrow{P_1}(s)B_1=-\overrightarrow{P_0}(0)$$
$$i.e.\overrightarrow{P_0}(s)\,(A0-sI)+\overrightarrow{P_1}(s)B_1=\vec{e},\ where\ \vec{e}=\left[|-1,0\right], \tag{7}$$

$$\overrightarrow{P_0}(s)C+\overrightarrow{P_1}(s)(A_1-sI)+\overrightarrow{P_2}(s)B_2=0,$$
$$\overrightarrow{P_1}(s)C+\overrightarrow{P_2}(s)\,(A_2-sI)+\overrightarrow{P_3}(s)B_3=0,$$
$$\overrightarrow{P_2}(s)C+\overrightarrow{P_3}(s)\,(A_3-sI)+\overrightarrow{P_4}(s)B_4=0,$$
$$\vdots$$
$$\vec{P}_{c-2}(s)C+\vec{P}_{c-1}(s)(A_{c-1}-sI)+\vec{P}_c(s)B_c=0,$$
$$\vec{P}_{c-1}(s)C+\vec{P}_c(s)(A-sI)+\vec{P}_{c+1}(s)B=0,$$
$$\vec{P}(s)C+\vec{P}_{c+1}(s)(A-sI)+\vec{P}_{c+2}(s)B=0, \tag{8}$$
$$\vdots$$

$$\therefore for\ 1\le n\le c-1$$
$$\vec{P}_{n-1}(s)C+\vec{P}_n(s)(A_n-sI)+\vec{P}_{n+1}(s)B_{n+1}=0, \tag{9}$$

and:

$$for\ n=c,c+1,c+2,..,$$
$$\vec{P}_{n-1}(s)C+\vec{P}_n(s)(A-sI)+\vec{P}_{n+1}(s)B=0 \tag{10}$$

Lemma 1 [24], the quadratic matrix equation related to equations represented by (10) is:

$$R^2(s)B+R(s)(A-sI)+c=0 \tag{11}$$

has the minimal non-negative solution given by:

$$(s)=\begin{bmatrix} r_{11}(s) & r_{12}(s) \\ 0 & r_{22}(s)\end{bmatrix} \tag{12}$$

To find $r_{11}(s)$, $r_{22}(s)$, $r_{12}(s)$, substituting $R(s)$ into (11) which proceeds as:

$$R(s)R(s)B+R(s)(A-sI)+C=0$$
$$\Rightarrow\begin{bmatrix} r_{11}(s) & r_{12}(s) \\ 0 & r_{22}(s)\end{bmatrix}\begin{bmatrix} r_{11}(s) & r_{12}(s) \\ 0 & r_{22}(s)\end{bmatrix}\begin{bmatrix} \xi & 0 \\ 0 & c\mu\end{bmatrix}$$
$$+\begin{bmatrix} r_{11}(s) & r_{12}(s) \\ 0 & r_{22}(s)\end{bmatrix}\left(\begin{bmatrix} -(\lambda+\xi+\gamma) & \gamma \\ 0 & -(\lambda+c\mu)\end{bmatrix}-\begin{bmatrix} s & 0 \\ 0 & s\end{bmatrix}\right)+\begin{bmatrix} \lambda & 0 \\ 0 & \lambda\end{bmatrix}$$
$$=\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$$
$$\Rightarrow r_{11}(s)=\frac{-(-(\lambda+\xi+\gamma+s))\pm\sqrt{(-(\lambda+\xi+\gamma+s))^2-4.\xi.\lambda}}{2\xi},$$
$$-(-(\lambda+c\mu+s))\pm p(-(\lambda+c\mu+s))^2-4.c\mu.\lambda$$
$$r_{22}(s)=2c\mu$$

Also $r_{12}(s)[c\mu(r_{11}(s)+r_{22}(s))-(\lambda+c\mu+s)]\ =\ -\gamma r_{11}(s)$

For minimal solution:

$$\Rightarrow r_{11}(s)=\frac{(\lambda+\xi+\gamma+s)-\sqrt{(\lambda+\xi+\gamma+s)^2-4\lambda\xi}}{2\xi},$$
$$r_{22}(s)=\frac{(\lambda+c\mu+s)-\sqrt{(\lambda+c\mu+s)^2-4c\mu}}{2c\mu}\ and$$
$$r_{12}(s)=\frac{\gamma r_{11}(s)}{-c\mu r_{11}(s)+c\mu\left(\frac{(\lambda+c\mu+s)+\sqrt{(\lambda+c\mu+s)^2-4c\lambda\mu}}{2c\mu}\right)}$$
$$Or, r_{12}(s)=\frac{\gamma r_{11}(s)}{-c\mu r_{11}(s)+c\mu r(s)_1};$$
$$where, r_1(s)=\frac{(\lambda+c\mu+s)+\sqrt{(\lambda+c\mu+s)^2-4c\lambda\mu}}{2c\mu}$$

$$Let\ r_{11}(s)=m_0(s)\ and\ r_{22}(s)=r_0(s), Then,$$
$$R(s)=\begin{bmatrix} r_{11}(s) & r_{12}(s) \\ 0 & r_{22}(s)\end{bmatrix}=\begin{bmatrix} m_0(s) & \frac{\gamma m_0(s)}{c\mu(r_1(s)-m_0(s))} \\ 0 & r_0(s)\end{bmatrix}$$

DOI: 10.3844/jmssp.2022.36.48

where,

$$m_0(s)=\frac{(\lambda+\xi+\gamma+s)-\sqrt{(\lambda+\xi+\gamma+s)^2-4\lambda\xi}}{2\xi}$$

$$r_0(s)=\frac{(\lambda+c\mu+s)-\sqrt{(\lambda+c\mu+s)^2-4c\mu}}{2c\mu}$$

$$r(s)_1=\frac{(\lambda+c\mu+s)+\sqrt{(\lambda+c\mu+s)^2-4c\lambda\mu}}{2c\mu}$$

Again:

$$R^2(s)=R(s)R(s)$$

$$\Rightarrow R^2(s)=\begin{bmatrix} m_0^2(s) & \dfrac{m_0\left((s)+r_0(s)\right)}{c\mu\left(r_1(s)-m_0(s)\right)} \\ 0 & r_0^2(s) \end{bmatrix}$$

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$$R^2(s)=R(s)R(s)$$

$$R^k(s)=\begin{bmatrix} m_0^k(s) & \dfrac{\gamma m_0(s)\left(\sum_{i=1}^{k} m_0^{k-1}(s)\right)}{c\mu\left(r_1(s)-m_0(s)\right)} \\ 0 & r^k(s) \end{bmatrix}, \text{ for all } k=1,2,3,\ldots \quad (13)$$

Lemma 2 [24], the matrix equation related to the equations represented by (9) are the following recurrence relation:

$$C+R_n(s)(A_n-sI)+R_n(s)R_{n+1}(s)B_{n+1}=0; 1\le n\le c-1$$

$$If\ \frac{\lambda}{c\mu}<1, let\ R_n(s)=\begin{bmatrix} r_{11}^n(s) & r_{12}^n(s) \\ 0 & r_{22}^n(s) \end{bmatrix}; 1\le n\le c-1 \quad (14)$$

satisfy the recurrence relation (14) then, we obtain $r_{11}{}^n(s)$, $r_{12}{}^n(s)$, and $r_{22}{}^n(s)$.

For $n=c-1, let\ R_{c-1}(s)=\begin{bmatrix} r_{11}^{c-1}(s) & r_{12}^{c-1}(s) \\ 0 & r_{22}^{c-1}(s) \end{bmatrix}$ then, the recurrence relation (14) becomes:

$$C+R_{c-1}(s)\left(A_{c-1}-sI\right)+R_{c-1}(s)R_c(s)B_c=0$$

$$\Rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}+\begin{bmatrix} r_{11}^{c-1}(s) & r_{12}^{c-1}(s) \\ 0 & r_2^{c-1}(s) \end{bmatrix}\left(\begin{bmatrix} -(\lambda+\xi+\gamma) & \gamma \\ 0 & -(\lambda+(c-1)\mu \end{bmatrix}\right)$$

$$-\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}+\begin{bmatrix} r_{11}^{c-1}(s) & r_{12}^{c-1}(s) \\ 0 & r_{22}^{c-1}(s) \end{bmatrix}\begin{bmatrix} r_{11}^{c}(s) & r_{12}^{c}(s) \\ 0 & r_{22}^{c}(s) \end{bmatrix}\begin{bmatrix} \xi & 0 \\ 0 & c\mu \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow r_{11}^{c-1}(s)=\frac{\lambda}{\lambda+\xi+\gamma+s-\xi r_{11}^{c}(s)},$$

$$r_{22}^{c-1}(s)=\frac{\lambda}{\lambda+\xi+\gamma+s-c\mu\, r_{22}^{c}(s)}$$

$$r_{22}^{c-1}(s)=\frac{\left(\gamma+c\mu r_{12}^{c}(s)\right)r_{11}^{c-1}}{\lambda+\xi+\gamma+s-c\mu\, r_{22}^{c}(s)}$$

and

Assuming $R_c(s) = R(s)$, it is seen that:

$$r_{11}^{c}(s)=m_0(s), r_{12}^{c}(s)=\frac{\gamma m_0(s)}{c\mu\left(r_1(s)-m_0(s)\right)}=k_0(s), r_{22}^{c}(s)=r_0(s)$$

and thus:

$$r_{11}^{c-1}(s)=\frac{\lambda}{\lambda+\xi+\gamma+s-\xi m_0(s)},$$

$$r_{22}^{c-1}(s)=\frac{\lambda}{\lambda+\xi+\gamma+s-c\mu r_0(s)}$$

$$r_{12}^{c-1}(s)=\frac{\left(\gamma+c\mu r_{12}^{c}(s)\right)r_{11}^{c-1}(s)}{\lambda+\xi+\gamma+s-c\mu r_0(s)}$$

and

Therefore:

$$R_{c-1}(s)=\begin{bmatrix} \dfrac{\lambda}{\lambda+\xi+\gamma+s-\xi m_0(s)} & \dfrac{\left(\gamma+c\mu r_{12}^{c}(s)\right)r_{11}^{c-1}(s)}{\lambda+(c-1)\mu+s-c\mu r_0(s)} \\ 0 & \dfrac{\lambda}{\lambda+\xi+\gamma+s-c\mu r_0(s)} \end{bmatrix} \quad (15)$$

is completely determined

For $n=c-2, let\ R_{c-2}(s)=\begin{bmatrix} r_{11}^{c-1}(s) & r_{12}^{c-2}(s) \\ 0 & r_{22}^{c-2}(s) \end{bmatrix}$ then, the recurrence relation (14) becomes:

$$C+R_{c-2}(s)(A_{c-2}-sI)+R_{c-2}(s)R_{c-1}(s)B_{c-1}=0$$

$$\Rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$+\begin{bmatrix} r_{11}^{c-2}(s) & r_{12}^{c-2}(s) \\ 0 & r_{22}^{c-2}(s) \end{bmatrix}\begin{bmatrix} r_{11}^{c-1}(s) & r_{12}^{c-2}(s) \\ 0 & r_{22}^{c-1}(s) \end{bmatrix}\begin{bmatrix} \xi & 0 \\ 0 & (c-1)\mu \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow r_{11}^{c-2}(s)=\frac{\lambda}{\lambda+\xi+\gamma+s-\xi r_{11}^{c-1}(s)},$$

$$r_{22}^{c-2}(s)=\frac{\lambda}{\lambda+\xi+\gamma+s-(c-1)\mu r_{22}^{c-2}(s)}$$

$$r_{12}^{c-2}(s)=\frac{\left(\gamma+c\mu r_{12}^{c-1}(s) r_{22}^{c-2}(s)\right)}{\lambda+(c-2)\mu+s-(c-1)\mu r_{22}^{c-1}(s)}$$

and

Therefore, $r_{11}^{c-2}(s), r_{22}^{c-2}(s)$ and $r_{12}^{c-2}(s)$ are obtained in terms of $m_0(s)$, $r_0(s)$, $r_1(s)$ for:

$$R_{c-2}(s)=\begin{bmatrix} r_{11}^{c-2}(s) & r_{12}^{c-2}(s) \\ 0 & r_{22}^{c-2}(s) \end{bmatrix}$$

Similarly, $R_{c-3}(s)$, $R_{c-4}(s)$, $\cdots$, $R_1(s)$ can be recursively determined in terms of $m_0(s)$, $r_0(s)$, and $r_1(s)$ which are already known.

Theorem 1 *If* $\frac{\lambda}{c\mu} < 1$, the Laplace transform of the transient state probabilities of the model under consideration [24], are:

$$\vec{\hat{P}}_n(s) = \vec{\hat{P}}_{c-1}(s) R^{n-c+1}(s); n \geq c \tag{16}$$

and:

$$\vec{\hat{P}}_n(s) = \vec{\hat{P}}_{n-1}(s) R_n(s) = \vec{\hat{P}}_0(s) R_n^*(s); 1 \leq n \leq c-1 \tag{17}$$

where, $R_n^*(s) = R_n(s) R_{n-1}(s) \cdots .R_1(s)$ because the relation (16) satisfies the Eq. (10) and the relation (17) satisfies the equation (9), which is shown in the following:

For $n \geq c$, substituting (16) in (10):

$$\vec{\hat{P}}_{n-1}(s) C + \vec{\hat{P}}_n(s)(A - sI) + \vec{\hat{P}}_{n+1}(s) B = 0$$

Again, for $1 \leq n \leq c-1$, substituting the relation (17) in Eq. (9):

$$\vec{\hat{P}}_{n-1}(s) C + \vec{\hat{P}}_n(s)(A - sI) + \vec{\hat{P}}_{n+1}(s) B = 0$$

Therefore, it is verified that $\hat{P}_n(s)$ expressed in (16) and (17) satisfies the governing system of differential equations in the Laplace domain as represented by equations (9) to (10). Hence from Eq. (7):

$$\vec{\hat{P}}_0(s) = \vec{\hat{P}}_0(0)\left[sI - A_0 - R_1(s) B_1\right]^{-1} \tag{18}$$

where A_0, B_1 is known and $R_1(s)$ can be recursively determined from Lemma 2.

Thus, the transient state probabilities of the model under consideration are given by:
and:

$$\vec{\hat{P}}_0(s) = \vec{\hat{P}}_{n-1}(s) R_n(s) = \vec{\hat{P}}_0(s) R_n^*(s), 1 \leq n \leq c-1$$
$$\vec{\hat{P}}_n(s) = \vec{\hat{P}}_{c-1}(s) R^{n-c+1}(s) = \vec{\hat{P}}_0(s) R_{c-1}^*(s) R^{n-c+1}(s), n \geq c$$

Where, $\hat{P}_0(s)$ is given by equation (18); $R^k(s)$, $\forall k$ is given by equation (13) and $R_n^*(s)$, for $n = 1,2,\cdots$ are recursively determined using Lemma 2. From (18),

$$\begin{aligned} P_{0,0}(s) &= \frac{s+\lambda+\eta-\mu r_{22}^1(s)}{\left(s+\lambda+\gamma-\xi r_{11}^1(s)\right)\left(s+\lambda+\gamma-\xi r_{22}^1(s)\right)-\eta\left(\gamma+\mu r_{12}^1(s)\right)} \\ P_{1,0}(s) &= \frac{s+\lambda+\eta-\mu r_{22}^1(s)}{\left(s+\lambda+\gamma-\xi r_{11}^1(s)\right)\left(s+\lambda+\gamma-\xi r_{22}^1(s)\right)-\eta\left(\gamma+\mu r_{12}^1(s)\right)} \end{aligned} \tag{19}$$

From the recurrence relation (17):

$$P_{0,n}(s) = P_{0,n-1}(s); 1 \leq n \leq c-1 \tag{20}$$

and:

$$\vec{\hat{P}}_{1,n}(s) = \hat{P}_{0,n-1}(s) r_{12}^n(s) r_{22}^n(s) 1 \leq n \leq c-1 \tag{21}$$

Now, using the relation (20):

$$\sum_{n=1}^{c-1} \hat{P}_{0,n}(s) = \hat{P}_{0,0}(s) \sum_{n=1}^{c-1} \prod_{i=1}^{n} r_{11}^i(s) \tag{22}$$

From the recurrence relation (16):

$$\begin{aligned} &\hat{P}_{0,n}(s) = \hat{P}_{c-1}(s)\left(m_0(s)\right)^{n-1+c} n \geq c \\ &\hat{P}_{1,n}(s) = \hat{P}_{c-1}(s) \frac{\gamma m_0(s) \sum_{i=1}^{n-c+1} (m_0(s))^{n-c+1} (r_0(s))^{i-1}}{c\mu r_1(r_1(s) - m_0(s))} + \hat{P}_{1,c}(s)(r_0(s))^{n-c+1} \end{aligned} \tag{23}$$

$$; n \geq c \tag{24}$$

Now, using relation (23):

$$\sum_{n=c}^{\infty} \hat{P}_{0,n}(s) = \hat{P}_{0,0}(s) \prod_{i=1}^{c-1} r_{11}^i(s) \sum_{n=c}^{\infty} \left(m_0(s)\right) n - c + 1 \tag{25}$$

Adding the relation (22) and (25):

$$\begin{aligned} &P^0(s) = \sum_{n=1}^{c-1} \hat{P}_{0,n}(s) + \sum_{n=c}^{\infty} \hat{P}_{0,n}(s) \\ &\Rightarrow P^0(s) = \hat{P}_{0,0}(s)\left(\sum_{n=1}^{c-1}\sum_{i=1}^{n} r_{11}^i(s) + \prod_{i=1}^{c-1}(s) \sum_{n=c}^{\infty}\left(m_0(s)\right)^{n-c+1}\right) \end{aligned} \tag{26}$$

Using the relation (21):

$$\sum_{n=1}^{c-1} \hat{P}_{1,n}(s) = \hat{P}_{1,1}(s) + \hat{P}_{1,2}(s) + \ldots + \hat{P}_{1,c-1}(s) \tag{27}$$

Again, using the relation (24), taking the sum from $n = c$ to ∞ and add it with (27):

$$\begin{aligned} \sum_{n=1}^{\infty} \hat{P}_{1,n}(s) &= \sum_{n=1}^{c-1} \hat{P}_{1,n}(s) + \hat{P}_{0,0}(s) \prod_{i=1}^{c-1} r_{11}^i(s) \frac{\gamma m_0(s)}{c\mu\left(r_1(s) - m_0(s)\right)} \\ &\times \sum_{n=c}^{\infty} \sum_{i=1}^{n-c+1} \left(m_0(s)\right)^{n-c+1-i} \left(r_0(s)\right)^{i-1} \\ &+ \hat{P}_{1,0}(s) \prod_{i=1}^{c-1} r_{22}^i(s) \sum_{n=c}^{\infty} \left(r_0(s)\right)^{n-c+1} \end{aligned} \tag{28}$$

The inverse transform $P(t)$ is given by the well-known inversion formula, by Abate and Whitt [27]:

$$\begin{aligned}P(t) &= \left(\frac{1}{2\pi i}\right)\int_{a-i\infty}^{a+i\infty} e^{st}\hat{P}(s)\,ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(a+iu)t}\hat{P}(a+iu)\,du\,[\because s = a+iu, a\,\text{complex number}] \\ &= \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty}(\cos ut + i\sin ut)\hat{P}(a+iu)\,du \\ &= \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty}\left[\mathrm{Re}\left(\hat{P}(a+iu)\right)\cos ut - \mathrm{Im}\left(P(a+iu)\right)\sin ut\right]du\end{aligned} \tag{29}$$

Equation (29) can also be replaced by cosine transform pair:

$$\mathrm{Re}\left\{\hat{P}(a+iu)\right\} = \int_0^{\infty} e^{-at}P(t)\cos ut\,dt \tag{30}$$

$$P(t) = \frac{2e^{at}}{\pi}\int_0^{\infty}\mathrm{Re}\left(\hat{P}(a+iu)\right)\cos ut \tag{31}$$

or by the sine transform pair:

$$\mathrm{Im}\left\{\hat{P}(a+iu)\right\} = -\int_0^{\infty} e^{-at}P(t)\sin ut\,dt \tag{32}$$

$$P(t) = -\frac{2e^{at}}{\pi}\int_0^{\infty}\mathrm{Im}\left(\hat{P}(a+iu)\right)\sin ut\,du \tag{33}$$

Applying the trapezoidal rule to the Laplace transform inversion formula in (31), Abate and Whitt [26]:

$$\begin{aligned}P(t) &= \frac{2e^{at}}{\pi}\int_0^{\infty}\mathrm{Re}\left(\hat{P}(a+iu)\right)\cos ut \\ &\approx P_h(t) \equiv \frac{he^{at}\,\mathrm{Re}\left(\hat{P}(a)\right)}{\pi} + \frac{2he^{at}}{\pi}\sum_{n=1}^{\infty}\mathrm{Re}\left(\hat{P}(a+inh)\right)\cos(nht).\end{aligned} \tag{34}$$

Eliminate the cosine term in (34) and produce nearly an alternating series by letting $h = \frac{\pi}{2t}$.

Letting $a = \frac{A}{2t}$ at the same time, Abate and Whitt [26]:

$$\begin{aligned}P_h(t) &= \frac{e^{A/2}}{2t}\mathrm{Re}\left(\hat{P}\left(\frac{A}{2t}\right)\right) + \frac{e^{A/2}}{t}\sum_{n=1}^{\infty}(-1)^n\,\mathrm{Re}\left(\hat{P}\left(\frac{A+2k\pi i}{2t}\right)\right) \\ Or, P_h(t) &= \left(\frac{e^{at}}{t}\right)\left[\frac{1}{2}\mathrm{Re}\left(\hat{P}(a)\right) + \sum_{n=1}^{\infty}\mathrm{Re}\left\{\hat{P}\left(a+\frac{n\pi i}{t}\right)\right\}(-1)^n\right]\end{aligned} \tag{35}$$

With the discretization error, Abate and Whitt [26]:

$$e_d \equiv e_d(P,t,A) = \sum_{n=1}^{\infty} e^{-nA}P\left((2n+1)t\right). \tag{36}$$

Which is called the Poisson summation formula with the bounds for two cases: first, if $|P(t)| \le M$, (i.e., the discretization error independent of the distribution) then, $|e_d| \le Me^{-A}/(1-e^{-A}) \approx 3Mte^{-A}$. Second, if $|P(t)| \le Mt$, then:

$$\left|e_d\right| \le \sum_{n=1}^{\infty} e^{-nA}(2n+1)Mt \le Mt\frac{3e^{-A}-e^{-2A}}{\left(1-e^{-A}\right)} \approx 3Mte^{-A} \tag{37}$$

Now, replace $P^0(s)$ from (26) and $P^1(s)$ from (28) in (35) to obtain the approximate value of $P_{0,n}(t)$ and $P_{1,n}(t)$. Also, replace $P^0(s)$ from (26) and $P^1(s)$ from (28) in (36) for the error of probability computation.

The formulas to find the expected number of customers waiting in the queue, the expected number of customers in the system, meantime spent waiting in queue, mean time spent in the system are respectively given in the following equations:

$$L_q(t) = \sum_{n=c}^{\infty}(n-c)\times P_{0,n}(t) + \sum_{n=c}^{\infty}(n-c)\times P_{1,n}(t) \tag{38}$$

$$L_s(t) = \sum_{n=1}^{\infty} n\times P_{0,n}(t) + \sum_{n=1}^{\infty} n\times P_{1,n}(t) \tag{39}$$

$$W_q(t) = \frac{L_q(t)}{\lambda(t)} \tag{40}$$

$$W_s(t) = \frac{L_s(t)}{\lambda(t)} \tag{41}$$

Numerical Interpretation

Figures (2) and (7) show that the higher the value of arrival rate λ, above is the graph which is quite natural with real-life situations. Also, we observe in the same graphs that as the time passes on, the expected number of customers in the queue ($Lq(t)$) as well as in the system ($L_s(t)$) are decreasing because there have state-dependent service rates provisioned in the model under study. Figures (3), (8), (13), (18), reveal that $L_q(t)$, $L_s(t)$, $W_q(t)$, $W_s(t)$ are decreasing with the increase of service rates, as it should be. Figures (4), (9), (14), (19) explore that $L_q(t)$, $L_s(t)$, $W_q(t)$, $W_s(t)$ decrease as the values of γ increases. This can be explained by the fact that the increase in the vacation rate leads to an increase in the probability of a busy period during which a significant number of customers can be served.

Figure (5), (10), (15), (20) depict that lower is the rate of waiting of server, the below is the graphs which imply that $L_q(t)$, $L_s(t)$, $W_q(t)$, $W_s(t)$ decrease faster when we decrease the rate η from 0.1 to 0.9. Figure (6), (11), (16), (21) suggest that $L_q(t)$, $L_s(t)$, $W_q(t)$, $W_s(t)$ decrease with time. This decrement is insignificant with the increase of impatience rate ξ from 0.01 to 0.9.

DOI: 10.3844/jmssp.2022.36.48

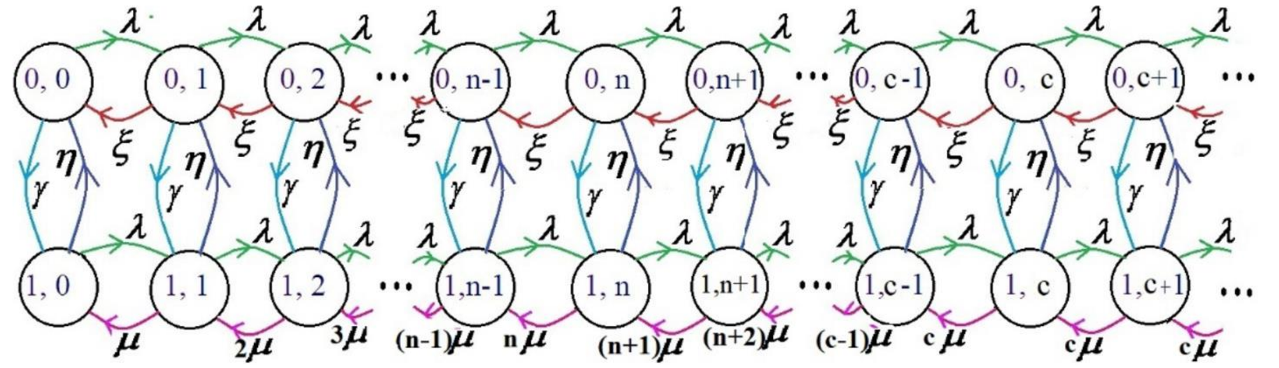

Fig. 1: State rate transition diagram

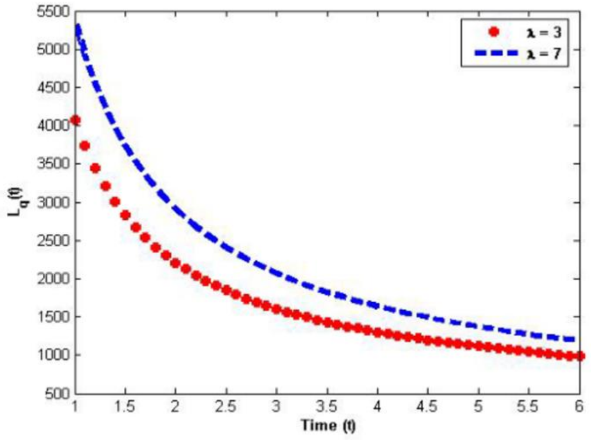

Fig. 2: Time (t) versus mean number of customer in the queue (L_q) for different mean arrival rate λ and $\mu = 4$, $\gamma = 0.5$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $N = 100$

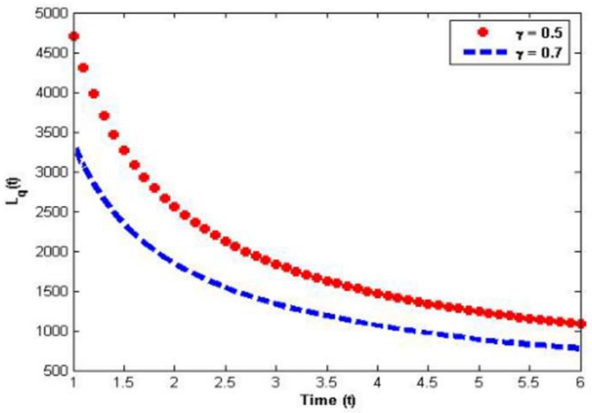

Fig. 4: Time (t) versus mean number of customer in the queue (L_q) for different vacation rate γ and $\lambda = 5$, $\mu = 4$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $N = 100$

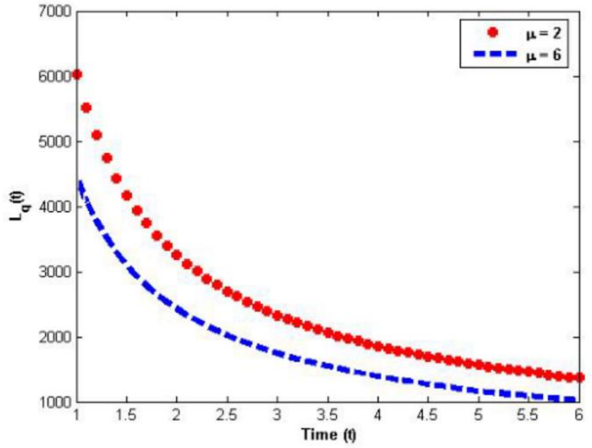

Fig. 3: Time (t) versus mean number of customer in the queue (L_q) for different mean service rate μ and $\lambda = 5$, $\gamma = 0.5$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $N = 100$

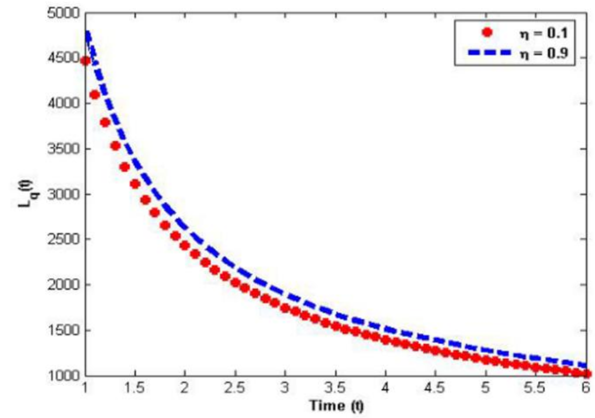

Fig. 5: Time (t) versus mean number of customer in the queue (L_q) for different servers' waiting rate η and $\lambda = 5$, $\mu = 4$, $\gamma = 0.5$, $\xi = 0.01$, $c = 5$, $N = 100$

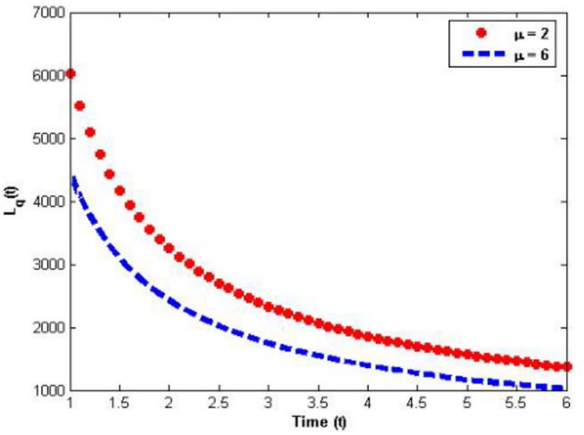

Fig. 6: Time (t) versus mean number of customer in the queue (L_q) for different impatience rate ξ and $\lambda = 5$, $\mu = 4$, $\gamma = 0.5$, $\eta = 0.6$, $c = 5$, $N = 100$

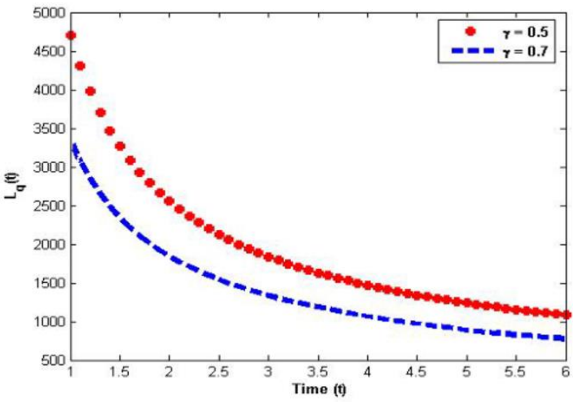

Fig. 7: Time (t) versus mean number of customer in the system (L_s) for different mean arrival rate λ and $\mu = 4$, $\gamma = 0.5$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $N = 100$

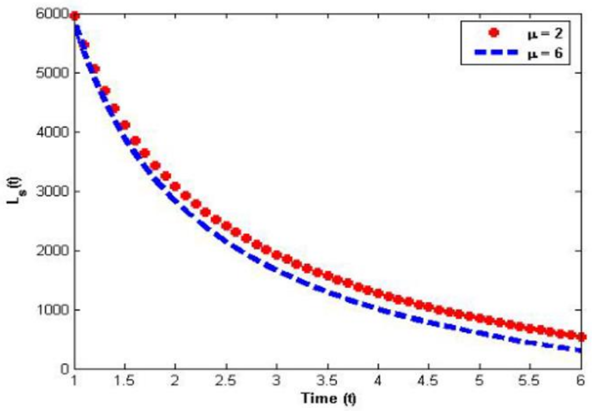

Fig. 8: Time (t) versus mean number of customer Fig. 12 Time (t) versus mean time spent in the in the system (Ls) for different mean service rate queue ($Wq(t)$) for different arrival rate λ and $\mu = 4$, μ and $\lambda = 3$, $\gamma = 0.5$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $\gamma = 0.5$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $N = 100$. $N = 100$

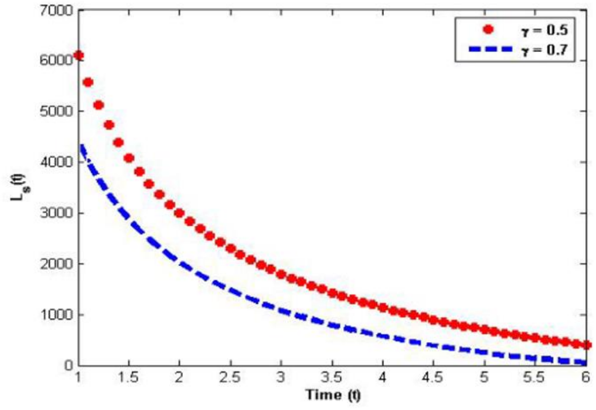

Fig. 9: Time (t) versus mean number of customer in the system (Ls) for different vacation rate γ and
$\lambda = 5$, $\mu = 4$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $N = 100$

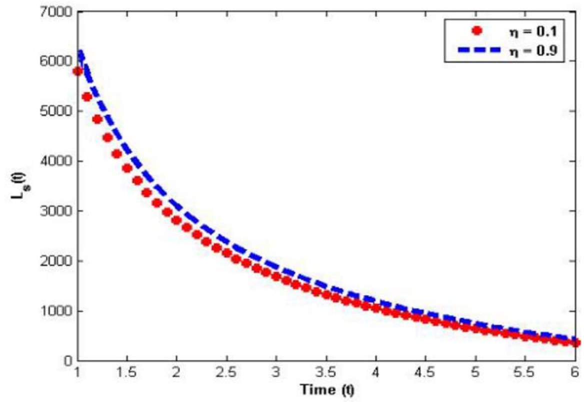

Fig. 10: Time (t) versus mean number of customer in the system (Ls) for different server's waiting rate η and $\lambda = 5$, $\mu = 4$, $\gamma = 0.5$, $\xi = 0.01$, $c = 5$, $N = 100$

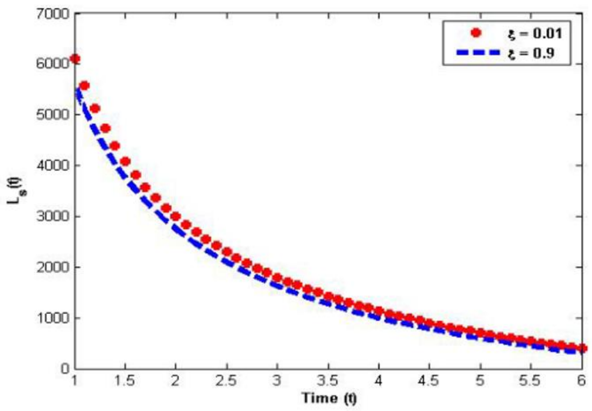

Fig. 11: Time (t) versus mean number of customer in the system (Ls) for different impatience rate ξ and $\lambda = 5$, $\mu = 4$, $\gamma = 0.5$, $\eta = 0.6$, $c = 5$, $N = 100$

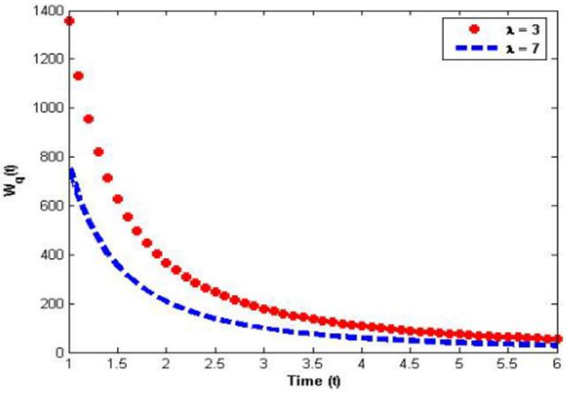

Fig. 12: Time (t) versus mean time spent in the queue (Wq(t)) for different arrival rate λ and μ = 4, γ = 0.5, η = 0.6, ξ = 0.01, c = 5, N = 100

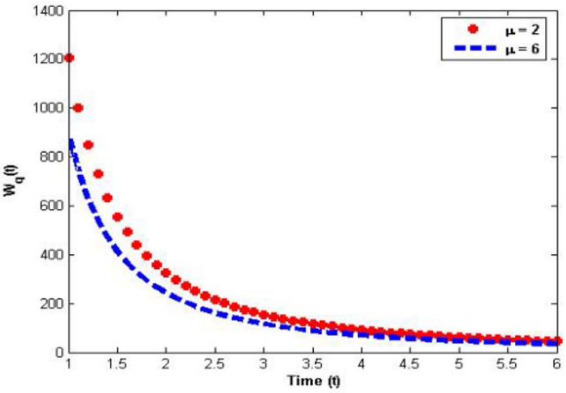

Fig. 13: Time (t) versus mean time spent in the Fig. 9 Time (t) versus mean number of customer queue ($W_q(t)$) for different service rate μ and λ =5, in the system (L_s) for different vacation rate γ and γ = 0.5, η = 0.6, ξ = 0.01, c = 5, N = 100. λ = 5, μ = 4, η = 0.6, ξ = 0.01, c = 5, N = 100

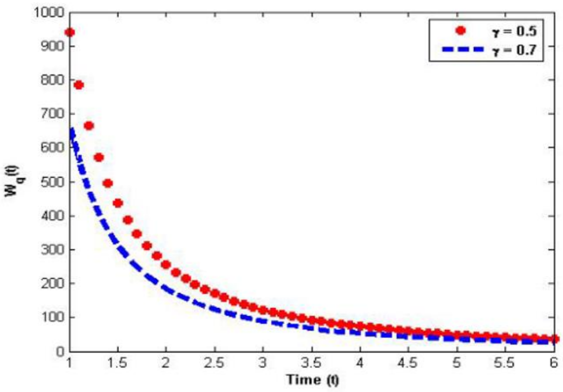

Fig. 14: Time (t) versus mean time spent in the Fig. 10 Time (t) versus mean number of customer queue ($W_q(t)$) for different vacation rate γ and λ = in the system (L_s) for different server's waiting rate 5, μ = 4, η = 0.6, ξ = 0.01, c = 5, N = 100. η and λ = 5, μ = 4, γ = 0.5, ξ = 0.01, c = 5, N = 100

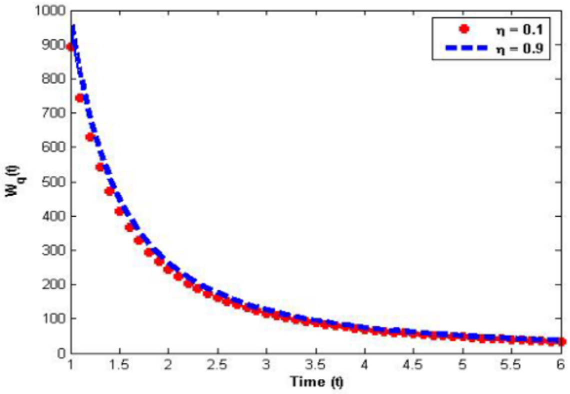

Fig. 15: Time (t) versus mean time spent in the queue ($W_q(t)$) for different server's waiting rate η Fig. 11 Time (t) versus mean number of customer and λ =5, μ=4, γ=0.5, ξ =0.01, c=5, N =100. in the system (L_s) for different impatience rate ξ and λ = 5, μ = 4, γ = 0.5, η = 0.6, c = 5, N = 100

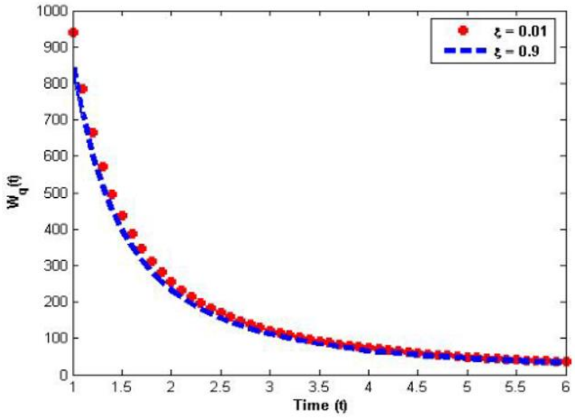

Fig. 16: Time (t) versus mean time spent in the queue ($W_q(t)$) for different impatience rate ξ and λ = 5,μ = 4, γ = 0.5, η = 0.6, c = 5, N = 100

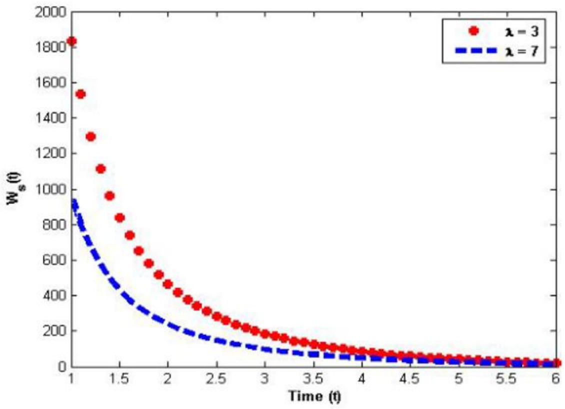

Fig. 17: Time (t) versus mean time spent in the system ($W_s(t)$) for different arrival rate λ and μ = 4, γ = 0.5, η = 0.6, ξ = 0.01, c = 5, N = 100

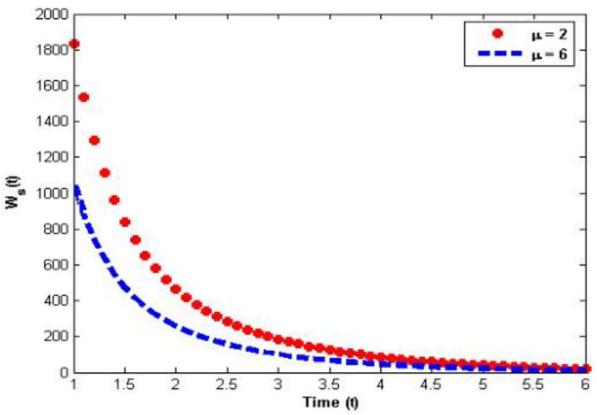

Fig. 18: Time (t) versus mean time spent in the system ($W_s(t)$) for different service rate μ and $\lambda = 5$, $\gamma = 0.5$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $N = 100$. 4.1 Table of Probability Errors

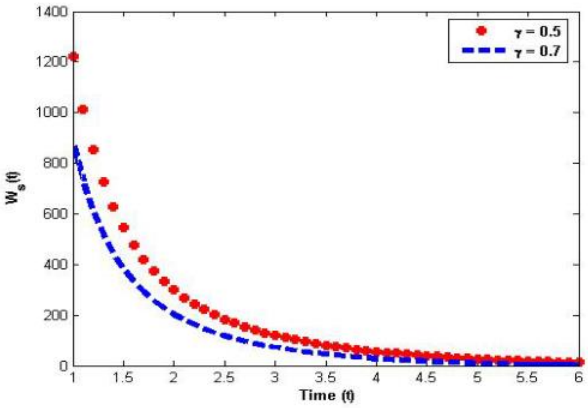

Fig. 19: Time (t) versus mean time spent in the system ($W_s(t)$) for different vacation rate γ and $\lambda = 5$, $\mu = 4$, $\eta = 0.6$, $\xi = 0.01$, $c = 5$, $N = 100$

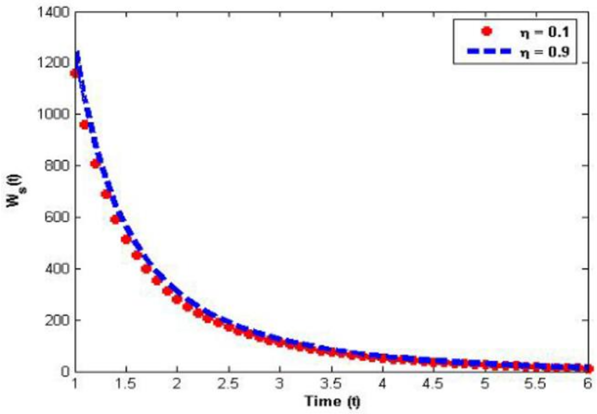

Fig. 20: Time (t) versus mean time spent in the system ($W_s(t)$) for different server's waiting rate η and $\lambda = 5$, $\mu = 4$, $\gamma = 0.5$, $\xi = 0.01$, $c = 5$, $N = 100$

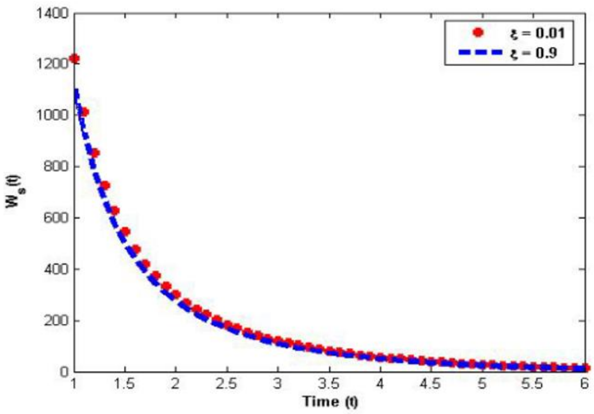

Fig. 21: Time (t) versus mean time spent in the system ($W_s(t)$) for different impatience rate ξ and $\lambda = 5$, $\mu = 4$, $\gamma = 0.5$, $\eta = 0.6$, $c = 5$, $N = 100$

Table of Probability Errors

The error of probability computation by using MATLAB programming is given in the following table, where E0 and E1 are the error during the numerical computation of $P_{0,n}(t)$ and $P_{1,n}(t)$ respectively.

Special Cases

Our derived results are different from the results obtained by, Ammar (2016; 2017) in the way that they had a single server queueing system and our model has multi-servers which gives complexity to the mathematical treatment. Our model is different from the model developed in Vijayashree and Janani (2015) in the way that their model did not take impatient customers and waiting for servers into account which we have. Hence our model is more general than their model.

Conclusion

The queueing model that we have developed in this investigation is more general than the queueing model studied by Ammar (2017) and Vijayashree and Janani (2015). We have set up the system of ordinary differential equations as balance equations of our model and have also derived the probability distribution functions explicitly in the Laplace domain. To find the probability distributions at any time t we have used the matrix geometric method, recursive and inverse Laplace numerical technique in the form of Fourier transform. The probability error table has also been obtained for both the server states. It has long been recognized that transient performance measures for the queues are complementary to the steady-state.

The model under study may have potential applications in automatic machining systems, multichannel telecommunications, security systems in the airport, train stations, and flexible manufacturing systems.

This model can also be further extended as future research under heterogeneous arrival and service rates with multiple vacation provisions which leads the problem much more realistic.

Authors Contributions

Ganesh Sapkota: Modelling of the mathematical structure of the problem and solving analytically and numerically. Actively engaged in reviewing the paper according to comments made by peer reviewers.

R. P. Ghimire: Concept design of the model and analyzing the problem from a practical point of view in terms of numerical results. Actively engaged in reviewing the paper according to comments made by peer reviewers. Interpreted the results so obtained in numerical forms.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues are involved

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