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Measurable Functional Calculi and Spectral Theory

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Abstract: In this article, the spectral theory is considered, we study the spectral families and their correspondence to the operators on the reflexive Banach spaces; assume *A* is a well-bounded operator on reflexive Lebesgue spaces then the operator *A* is a scalar type spectral operator. The main goals are to obtain the characterization of the well-bounded operators in the terms of the associated spectral family in the topology of dual pairing and to construct the continuous functional calculus for well-bounded operators on the Lebesgue space.

Keywords: Functional Calculus, Banach Space, Spectral Theorem, C*-Algebra, Measurable Space, Spectral Integral, Well-Bounded Operator

Introduction

This article is dedicated to the spectral theory of the operators that are defined on the subset of the reflexive Banach space *X*. An important example of such operators is a class of well-bounded operators, which have spectral decomposition with special properties. Let us presume that the functional calculus defined on the Banach algebra of the continuous functions *AC* ([*a*, *b*]) on a compact interval [*a*, *b*] then its operator is well-bounded. Assuming that the functional calculus of the well-bounded operator on L^p, $1 < p < \infty$ space is contractive then this operator has a scalar-type spectral. The last statement is not true in the cases when the Banach spaces are not reflexive, for example, on L^∞ (Budde and Landsman, 2016; Colombo *et al*., 2007; Haase, 2014).

Let us consider a simpler example of the theory in Banach space, the structure of the projection measure in the Hilbert space *H*. Let (Z, Σ, η) be a measurable Borel space and $\{H_z\}_{x\in z}$ be an η- measurable set of separable Hilbert spaces (Schmüdgen, 2012). The projection-valued measure *E* on (Z, Σ) can be defined as a mapping from Σ to the set of self-adjoint orthogonal projections on *H* that satisfies $E(Z) = Id_H$ and the mapping from σ- algebra Σ into the field $\phi \mapsto (E(\phi)x, y)$ is a complex measure on Σ. In terms of the functional calculus this definition can be reformulated in the following form: Let (Φ, *H*) be functional calculus on a measurable space (*Z*, Σ), the projection-valued measure is a mapping:

$$E:\Sigma \to L(H), \quad E(\phi) = \Phi(\chi_B) \in L(H)$$

for any $\phi \in \Sigma$.

The main result of the theory for separable Hilbert spaces is the statement that for each projection-valued measure on the measurable space there is a unique measurable functional calculus that generates this projection-valued measure, and conversely, for each measurable functional calculus on a measurable space, there is a uniquely defined projection-valued measure (Haase, 2014).

In the present article, these results are developed and extended in the case of the reflexive Banach spaces. We show that presuming (Φ, *X*) is a functional calculus on the measurable space (*Z*, Σ) then there is a semi-finite measure space (Ω, *F*, *μ*) and operator $U: X \to L^p$ ((Ω, *F*, *μ*) and an injective pointwise continuous ∗-homomorphism:

$$F: M(Z, \Sigma) \to M(\Omega, \mathsf{F}),$$

such that $\Phi(f) = UM_{Ff}U^{-1}$, where M_{Ff} is the operator of the multiplication by function *f*.

An important result of the representation theory is the following statement that if the *AC* functional calculus of the operator is contractive then the operator can be represented as the integral concerning a spectral measure.

There is extensive literature on the subject, for clarification of definitions and basic concepts, the reader can consult the list of references.

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The Spectral Decomposition for the Operator in Reflexive Banach Spaces

Some Definitions and Notations

The letter P denotes the scalar field usually real or complex numbers, the letters X, Y, Z denote reflexive Banach spaces; $L(X)$ denotes the Banach algebra of all bounded linear operators on X, for any real compact interval $AC([a, b])$ denotes the Banach algebra of all continuous functions with its natural norm and $BV([a, b])$ denotes the Banach algebra of all functions of bounded variation with its natural norm. It is easy to show that if a function f belongs to $AC([a, b])$ then this function f necessarily belongs to $BV([a, b])$, however not reciprocally, there is such function $g \in BV([a, b]$, which $g \notin BV([a, b]$, in other words, the algebra $AC([a, b])$ is a proper subalgebra of the algebra $BV([a, b])$. Indeed, let $f \in AC([a, b])$ then for any $\varepsilon > 0$ there is $\delta > 0$ such that for any sequence of disjointed intervals $\{(a_i, b_i)\}_{i=\mathbf{1},\ldots,n}$, the property: That from $\sum_{i=\mathbf{1},\ldots,n} b_i - a_i < \delta$ follows $\sum_{i=\mathbf{1},\ldots,n} \| f(b_i) - f(a_i) \| < \varepsilon$ is satisfied. Let us divide the interval $[a, b]$ by points $a = \lambda_1 < \lambda_2 < \ldots < \lambda_n = b$ into parts in such a way that $\lambda_{i+1} - \lambda_i < \delta$ for $i = 1,\ldots, n$-1. Then for any division $\{(\sigma_{j+1}, \sigma_j)\}_{j=\mathbf{1},\ldots,m-\mathbf{1}}$ of the interval $[\lambda_{i+1}, \lambda_i]$, on these parts, the sum $\sum_{j=\mathbf{1},\ldots,n-\mathbf{1}} \| f(\sigma_{j+\mathbf{1}}) - f(\sigma_j) \|$ $\sum_{j=\mathbf{1},\ldots,n-\mathbf{1}} \| f(\sigma_{j+\mathbf{1}}) - f(\sigma_j) \| < \varepsilon$ is, so the variation of the function f on the interval $[\lambda_{i+1}, \lambda_i]$ is necessarily less than ε, thus the variation of the function f on the interval $[a, b]$ is less than εn, so the function $f \in BV([a, b])$.

Definition 1.

Let $A: X \to Y$ be an operator defined on Banach spaces X then the operator $A^*: Y^* \to X^*$ is called the adjoint operator to $A: X \to Y$, namely, $(A^*(f))(x) = f(A(x))$ for all $f \in Y^*$ and all $x \in X$.

In particular, assuming X is a reflexive Banach space then if operator $A: X \to X$ then the adjoint operator is $A^*: X^* \to X^*$ if operator $A: X \to X^*$ then the adjoint operator is $A^*: X^* \to X^*$.

Definition 2.

Let operator $A: X \to X$ then the set $\rho(A)$ of all complex numbers such that:

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \, has \, inverse\}$$

is called the resolvent set.

The complement $\sigma(A)$ to the resolvent set is a spectrum of the operator $A: X \to X$.

The operator $R(\lambda, A) = (\lambda I - A)^{-1}$ is called a resolvent of operator A.

Definition 3.

The set $\{E(\lambda), \lambda \in \mathbb{R}\}$ of projection operators that satisfies the following conditions:
$E(\lambda)\ E(\mu) = E(\mu)\ E(\lambda) = E(\lambda)$ for $\lambda \leq \mu$; and $\sup_{\lambda} \| E(\lambda) \| < \infty$:

1. $E(\lambda) = strong - \lim_{\mu \to \lambda} E(\mu)$

and:

2. $strong - \lim_{\lambda \to -\infty} E(\lambda) = O$
$strong - \lim_{\lambda \to \infty} E(\lambda) = I$
3. $strong - \lim_{\lambda \to \infty} E(\lambda) = I$
4. $A = \int_{\mathbb{R}} \lambda \, dE(\lambda) = = strong - \lim_{N \to \infty} \int_{[-N, N]} \lambda \, dE(\lambda)$

is called the spectral family of operator A.

Condition 1 is a definition of the projection, which means the operator $E(\lambda)$is a projection onto the subspace of X created by all eigenvectors corresponding to all eigenvalues that are no larger than λ.

An operator A can be written as:

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda)$$

where, E is a spectral family of A, all limits are understood as limits concerning the natural topologies. This integral is an operator-valued Riemann-Stieltjes integral in the topology of the operator norm L.

Let us consider the integral $\int_{[a,b]} f(\lambda) \, dE(\lambda)$ as an operator-valued Riemann-Stieltjes integral. We can build a partition P of the compact interval $[a, b]$ as $a = \lambda_0 < \lambda_1 < \ldots < \lambda_n$ and the direction of the partition $|P| = \max_{i=\mathbf{1},\ldots,n} |\lambda_i - \lambda_{i-\mathbf{1}}|$ then if for any chosen set $\{\xi_i\}_{1,\ldots,n}$ of points (ξ_i) λ_{i-1}, λ_i there is a limit:

$$\lim_{|P| \to \mathbf{0}} \sum_{i=\mathbf{1},\ldots,n} f(\xi_i) \big(E(\lambda_i) - E(\lambda_{i-\mathbf{1}}) \big),$$

and this limit is independent of the specifics of the partitions, this limit is called the Riemann-Stieltjes integral of the continuous function f and can be written as:

$$\int_{[a,b]} f(\lambda)\, dE(\lambda) = \\ = \lim_{|P|\to 0} \sum_{i=1,\dots,n} f(\xi_i)\big(E(\lambda_i) - E(\lambda_{i-1})\big).$$

Theorem 1.

For the existence of the integral:

$$I = \int_{[a,b]} f(\lambda)\, dE(\lambda)$$

it is necessary and sufficient that:

$$\lim_{|P|\to 0} \sum_{i=1,\dots,n} \left(\sup_{\xi_i \in [\lambda_{i-1},\lambda_i]} f(\xi_i) - \inf_{\xi_i \in [\lambda_{i-1},\lambda_i]} f(\xi_i) \right) \times \\ \times \big(E(\lambda_i) - E(\lambda_{i-1})\big) = 0.$$

The proof of this theorem is rather standard: first is building the upper and lower Darboux-Stieltjes sums and finding their difference next showing that the conditions of the theorem are the necessary and sufficient conditions that the difference between the upper and lower Darboux-Stieltjes's sums converges to zero.

Theorem 2.

If the function f is continuous and $\|E(\lambda)\|$ belongs to $BV[a, b]$ as a function of λ then the integral $I = \int_{[a,b]} f(\lambda)\, dE(\lambda)$ exists.

This theorem is the consequence of theorem 1.

Theorem 3. If the function $f \in AC[a, b]$ then the integral $I = \int_{[a,b]} f(\lambda)\, dE(\lambda)$ exists.

Proof. For any $f \in AC[a, b]$ the mapping:

$$\Psi(f) = f(a)\, E(a) + \int_{[a,b]} f(\lambda)\, dE(\lambda)$$

is defined as the homomorphism $\Psi(f) : AC([a,b]) \to L(X)$ for which the following estimation:

$$\|\Psi(f)\| \le \sup_{\lambda\in[a,b]} \|E(\lambda)\| \left(|f(b)| + \operatorname*{var}_{[a,b]} f \right)$$

holds for all $f \in AC[a, b]$.

Lemma 1.

Let, $f \in AC[a, b]$ and let φ be a continuous function of the real argument t defined on $[a, b]$ then:

$$\mathit{Stieltjes} \int_{[a,b]} \varphi(t)\, d\, f(t) = \\ = \mathit{Lebesque} \int_{[a,b]} \varphi(t)\, f'(t)\, dt.$$

Proof of the existence of both integrals is obvious.

By definition, the Stieltes integral is the limit of the following integral sums:

$$\sum_{i=1,\dots,n} \varphi(\xi_i)\big(f(t_i) - f(t_{i-1})\big).$$

Since:

$$f(t_i) - f(t_{i-1}) = \int_{[t_{i-1},t_i]} f'(t) dt$$

we have:

$$\sum_{i=1,\dots,n} \varphi(\xi_i)\big(f(t_i) - f(t_{i-1})\big) - \int_{[a,b]} \varphi(t)\, f'(t)\, dt = \\ = \sum_{i=1,\dots,n} \int_{[t_{i-1},t_i]} \big(\varphi(\xi_i) - \varphi(t)\big) f'(t) dt$$

and:

$$\left| \sum_{i=1,\dots,n} \varphi(\xi_i)\big(f(t_i) - f(t_{i-1})\big) - \int_{[a,b]} \varphi(t)\, f'(t)\, dt \right| \le \\ \le \sum_{i=1,\dots,n} \left(\sup_{\xi\in[t_{i-1},t_i]} \varphi(\xi) - \inf_{\xi\in[t_{i-1},t_i]} \varphi(\xi) \right) \int_{[t_{i-1},t_i]} |f'(t)| dt.$$

Next, we have that $\sup_{i=1,\dots,n} \left(\sup_{\xi\in[t_{i-1},t_i]} \varphi(\xi) - \inf_{\xi\in[t_{i-1},t_i]} \varphi(\xi) \right)$ converge to zero when the maximal longitude of the segments of the partitions converges to zero. The lemma has been proven.

Theorem 4.

Let, X be a reflexive Banach space and let the operator $A \in L(X)$ be well-bounded then there is a unique spectral family $E(\cdot)$ in X such that:

$$A = a\, E(a) + \int_{[a,b]} \lambda\, dE(\lambda).$$

Remarks the spectral family $E(\cdot)$ is concentrated on a compact interval.

Proof.

Let us define a functional calculus $\Upsilon : AC([a,b]) \to LB(X)$. We define a set $F(\lambda, \eta)$ of all real-valued continuous functions $f \in AC([a, b])$ such that:

$$f = \begin{cases} 1 \quad on \quad [a,\lambda] \\ decreasing \quad on \quad [\lambda, \lambda+\eta] \\ 0 \quad on \quad [\lambda+\eta, b] \end{cases}$$

for all $\lambda \in [a, b)$ and $0 < \eta < (b-\lambda)$. Next, we have $\|f\|_{Bound} \le 1$ for any $f \in F(\lambda, \eta)$. The class $K(\lambda, \eta)$ can be defined as a closure in the weak topology:

$$K(\lambda,\eta) = \\ = weak\ cl\{\Upsilon(f) : f \in F(\lambda,\eta)\} \subset LB(X^*)$$

For $\eta_1 < \eta_2$ we obtain $K(\lambda, \eta_1) \subset K(\lambda, \eta_2)$ and it can bed educed that set $K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$ is a weakly compact uniformly bounded set.

The set Z is a subset of the reflexive Banach space defined by the formula:

$$Z(\lambda) = \\ = \left\{ x \in X : \Upsilon(f)x = 0, \quad f \in \bigcup_{\eta>0}\left(1 - F(\lambda,\eta)\right)\right\}. \\ y \in Z(\lambda) \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$$

Let then there is a net $\{g_\alpha\}_{\alpha\in\Lambda} \subset K(\lambda,\eta)$ with the following property:

$$\langle Ex, y^*\rangle = \lim_{\alpha\in\Lambda}\langle \Upsilon(g_\alpha)x, y^*\rangle = \\ = \lim_{\alpha\in\Lambda}\langle\left(1 - \Upsilon(1 - f_\alpha)\right)x, y^*\rangle$$

for all $x \in X$. Since $\langle Ex, y^*\rangle = \langle x, y^*\rangle$ we have $x \in Rang(E)$ thus set $Z(\lambda)$ is the range of each $K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$.

For any $\theta > 0$, there is $\eta_0 > 0$ such that $0 \le f(t) \le \frac{\theta}{2}$ for all $t \in [\lambda, \lambda + \eta_0]$, so for $E \in K(\lambda, \eta_0)$ there is a net $\{g_\alpha\}_{\alpha\in\Lambda} \subset F(\lambda, \eta_0)$ with the property $weak - \lim_{\alpha\in\Lambda}\Upsilon(g_\alpha) = E$.

Now, we are going to apply the fourth condition of the definition:

$$\int_{[a,b]}\left|(fg_\alpha)'\right| = \int_{[a,b]}\left|f'g_\alpha + f\ g_\alpha'\right| \le \\ \le \int_{[\lambda,\lambda+\eta_0]}\left|f'g_\alpha\right| + \int_{[\lambda,\lambda+\eta_0]}\left|+f\ g_\alpha'\right| \le \\ \le \frac{\theta}{2} + \frac{\theta}{2} = \theta,$$

so:

$$\left|\langle \Upsilon(f)x, x^*\rangle\right| \le \left|\langle\Upsilon(f)x, y^*\rangle\right| = \\ = \left|\langle \Upsilon(f)Ex, y^*\rangle\right| = \\ = \left|\langle Ex, (\Upsilon(f))^* y^*\rangle\right| = \\ = \left|\lim_{\alpha\in\Lambda}\langle \Upsilon(g_\alpha)x, (\Upsilon(f))^* y^*\rangle\right| = \\ = \left|\lim_{\alpha\in\Lambda}\langle \Upsilon(f\ g_\alpha)x, y^*\rangle\right| \le \\ \le \operatorname{sub}_{\alpha\in\Lambda}\|\Upsilon(f\ g_\alpha)\|\|x\|\|y^*\|$$

for all $y^* \in X^*$ so $E \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$. Thus, from the inequality $\left|\langle \Upsilon(f)x, y^*\rangle\right| \le \theta\|\Upsilon\|\|x\|\|y^*\|$ follows $\Upsilon(f)x = 0$, so the range of E coincides with $Z(\lambda)$; the set E is a projection.

Let us establish that $K(\lambda, \eta)$ is a commutative multiplicative semigroup. Let $\hat{K}, \breve{K} \in K(\lambda,\eta)$, us have that there are nets $\{g_\alpha\}_{\alpha\in\Lambda}, \{h_\beta\}_{\beta\in\mathrm{B}} \in F(\lambda,\eta)$ such that:

$$\hat{K} = weak - \lim_{\alpha\in\Lambda}\Upsilon(g_\alpha)$$

and:

$$\breve{K} = weak - \lim_{\beta\in\mathrm{B}}\Upsilon(h_\beta).$$

For all $x \in X$, we have:

$$\langle \hat{K}\ \breve{K}x, y^*\rangle = \lim_{\alpha\in\Lambda}\langle \Upsilon(g_\alpha)\breve{K}x, y^*\rangle = \\ = \lim_{\alpha\in\Lambda}\langle \breve{K}x, (\Upsilon(g_\alpha))^* y^*\rangle = \\ = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in\mathrm{B}}\langle \Upsilon(h_\beta)\,x, (\Upsilon(g_\alpha))^* y^*\rangle\right\} = \\ = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in\mathrm{B}}\langle \Upsilon(g_\alpha h_\beta)x, y^*\rangle\right\} = \\ = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in\mathrm{B}}\langle\ \Upsilon(h_\beta)\Upsilon(g_\alpha)x, y^*\rangle\right\} = \\ = \lim_{\alpha\in\Lambda}\langle \breve{K}\Upsilon(g_\alpha)x,\ y^*\rangle = \\ = \lim_{\alpha\in\Lambda}\langle \Upsilon(g_\alpha)x, (\breve{K})^* y^*\rangle = \\ = \langle \hat{K}\ x, (\breve{K})^* y^*\rangle = \langle\ \breve{K}\hat{K}x,\ y^*\rangle,$$

so $\hat{K}\ \breve{K} = \breve{K}\hat{K}$, thus:

$$E(\lambda) \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta),$$

uniqueness follows from the properties of the projections. We define the set of the projection $\{E(\lambda)\}_{\lambda\in[a,b]}$ on X by presuming $E(\lambda) = O$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda > b$.

Now, let us establish the properties $\{E(\lambda)\}_{\lambda\in[a,b]}$. Assuming that $a \le \lambda < \mu < b$ and assuming η is large enough, we are going to obtain that from $E(\lambda)$, $E(\mu) \in K(\lambda, \eta)$ follows $E(\lambda)$, $E(\mu) = E(\mu)\,E(\lambda) = E(\lambda)$. If $\eta = \mu - \lambda$, then from $E(\lambda) \in K(\lambda, \eta)$ follows the existence of the nets $\{g_\alpha\}_{\alpha\in\Lambda} \in F(\lambda,\eta)$ $\{h_\beta\}_{\beta\in\mathrm{B}} \in F(\lambda,\eta)$ and with the properties $weak - \lim_{\alpha\in\Lambda}\Upsilon(g_\alpha) = E(\lambda)$ and $weak - \lim_{\beta\in\mathrm{B}}\Upsilon(h_\beta) = E(\mu)$. Next, since $g_\alpha h_\beta = g_\alpha$ we have:

$$\begin{aligned}&\langle E(\lambda)\,E(\mu)x, y^*\rangle =\\&=\lim_{\alpha\in\Lambda}\langle \Upsilon(g_\alpha)\,E(\mu)x, y^*\rangle =\\&=\lim_{\alpha\in\Lambda}\langle\, E(\mu)x, (\Upsilon(g_\alpha))^* y^*\rangle =\\&=\lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\langle\, \Upsilon(h_\beta)x, (\Upsilon(g_\alpha))^* y^*\rangle\right\} =\\&=\lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\langle\, \Upsilon(g_\alpha)\Upsilon(h_\beta)x, y^*\rangle\right\} =\\&=\lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\langle\, \Upsilon(g_\alpha h_\beta)x, y^*\rangle\right\} =\\&=\lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\langle\, \Upsilon(g_\alpha)x, y^*\rangle\right\}\end{aligned}$$

for all $x \in X$, $y^* \in X^*$. Thus, it has been obtained $\langle E(\lambda)\,E(\mu)x, y^*\rangle = \langle E(\lambda)x, y^*\rangle$ and so equality of projection:

$$E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$$

holds for all a $\le \lambda < \mu < b$.

Since $strong - \lim_{\mu\to\lambda+0} E(\mu) = E(\lambda+0)$ we have $E(\lambda + 0) \in K(\lambda)$.

For any pair $x \in X$, $y^* \in X^*$ and for any function $f \in AC[a, b]$ the morphism $f \mapsto \langle\, \Upsilon(f)x, y^*\rangle$ is an element of the dual space to $AC([a, b])$ and since $AC[a, b]$ is isometric to $L^1\ [a, b]\oplus C$, from the duality argument, we have that there are $\gamma\langle x, y^*\rangle \in L^\infty([a,b])\tilde{c}\langle x, y^*\rangle \in C$, which satisfy the following equality:

$$\begin{aligned}&\langle \Upsilon(f)x, y^*\rangle =\\&=\tilde{c}\langle x, y^*\rangle f(b) + \int_{[a,b]} f'(t)\gamma\langle x, y^*\rangle(t)dt\end{aligned}$$

for all $f \in AC([a, b])$.

For any $\lambda \in [a, b)$, we assume $0 < \lambda + \eta < b$ then the function:

$$g(\lambda,\eta)(t) = \begin{cases}1 & on\quad [a,\lambda]\\ ecreasing & on\quad [\lambda, \lambda+\eta]\\ 0 & on\quad [\lambda+\eta, b]\end{cases}$$

belongs to $F(\lambda, \eta)$ and:

$$\langle \Upsilon(g(\lambda,\eta))x, y^*\rangle = -\frac{1}{\eta}\int_{[\lambda,\lambda+\eta]}\gamma\langle x, y^*\rangle(t)dt\,.$$

Thus, there is a weak limit $g(\lambda,\eta)\xrightarrow{weak-\eta\to 0+} E(\lambda)$.

So, λ - almost everywhere, we obtain $\gamma\langle x, y^*\rangle(\lambda) = -\langle E(\lambda)x, y^*\rangle$, and for arbitrary $x \in X$, $y^* \in X^*$, the integral equality:

$$\langle \Upsilon(f)x, y^*\rangle = \langle x, y^*\rangle f(b) - \int_{[a,b]} f'(\lambda)\langle E(\lambda)x, y^*\rangle d\lambda$$

holds for all $f \in AC[a, b]$.

Next, we have:

$$\begin{aligned}&\left\langle\left(\overset{\oplus}{\int_{[a,b]}} f\ dE\right)x, y^*\right\rangle =\\&=\lim_{\Lambda\in\Pi}\left\{\begin{aligned}&\langle E(b)x, y^*\rangle f(b) -\\&-\left\langle\sum_{\Lambda}(f(\lambda_i) - f(\lambda_{i-1}))E(\lambda_i)x, y^*\right\rangle\end{aligned}\right\} =\\&=\langle x, y^*\rangle f(b) -\\&-\lim_{\Lambda\in\Pi}\left\{\sum_{\Lambda}(f(\lambda_i) - f(\lambda_{i-1}))\langle E(\lambda_i)x, y^*\rangle\right\} =\\&=\langle x, y^*\rangle f(b) -\\&-\int_{[a,b]} f'(\lambda)\langle E(\lambda)x, y^*\rangle d\lambda = \langle \Upsilon(f)x, y^*\rangle.\end{aligned}$$

Thus, by taking $f(\lambda) = \lambda$, we have:

$$\langle Ax, y^*\rangle = b\langle x, y^*\rangle - \int_{[a,b]}\langle E(\lambda)x, y^*\rangle d\lambda\,.$$

The Characteristics of Well-Bounded Operators in Terms of the Weak Spectral Family

Definition 4.

The set $\{E(\lambda) \in L(X^*), \lambda \in \mathbb{R}\}$ of projection operators that satisfies the following conditions:

1. E(·) is concentrated on a compact interval $[a, b]$
2. $E(\lambda)$, $E(\mu) = E(\mu)\,E(\lambda) = E(\lambda)$ for $\lambda \le \mu$; and $\sup_\lambda\|E(\lambda)\| < \infty$
3. $E(\lambda) = O$ for all $\lambda < a$ and $E(\lambda) = I$ for all b $< \lambda$
4. there is $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{[t,t+\varepsilon]}\langle x, E(\lambda)y^*\rangle d\lambda = \langle x, E(t)y^*\rangle$

for all $x \in X$, y^*, X^* and for all $t \in (a, b)$
is called a weak spectral family.

Theorem 5. Let $A \in L(X)$ be a linear well-bounded operator then there is a unique weak spectral family $\{E(\lambda) \in L(X^*),\ \ \lambda \in \mathbb{R}\}$ concentrated on $[a, b]$ such that the equality:

$$\left\langle A(x), y^*\right\rangle = b\left\langle x, y^*\right\rangle - \int_{[a,b]} \left\langle x, E(\lambda) y^*\right\rangle d\lambda$$

holds for all $x \in X$, y^*, X^*.

Proof. Let Φ denotes a functional calculus $\Phi: AC([a,b]) \to LB(X)$ then we define a functional calculus $\Upsilon: AC([a,b]) \to LB(X^*)$ by the formula $\Upsilon(f) = (\Phi(f))^*$.

The Υ is compact functional calculus in the weak topology of $AC[a, b]$ for the operator $A^* \in LB(X^*)$.

Let us define a set $F(\lambda, \eta)$ of all real-valued functions $f \in AC([a, b])$ such that:

$$f = \begin{cases} 1 \quad on \quad [a, \lambda] \\ decreasing \quad on \quad [\lambda, \lambda+\eta] \\ 0 \quad on \quad [\lambda+\eta, b] \end{cases}$$

for all $\lambda \in [a, b)$ and $0 < \eta < (b-\lambda)$. The class $K(\lambda, \eta)$ can be defined as a closure in the weak topology:

$$K(\lambda,\eta) = \\ = weak\ cl\left\{\Upsilon(f) : f \in F(\lambda,\eta)\right\} \subset LB(X^*)$$

From $\eta_1 < \eta_2$ follows $K(\lambda,\eta_1) \subset K(\lambda,\eta_2)$ and we can deduce that set $K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$ is a weakly compact uniformly bounded set.

We define the subset Z of the reflexive Banach space by the formula:

$$Z(\lambda) = \\ = \left\{x^* \in X^* : \Upsilon(f)x^* = 0,\ f \in \bigcup_{\eta>0}\left(1 - F(\lambda,\eta)\right)\right\}.$$

Let $y^* \in Z(\lambda) \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$ then there is a net $\{g_\alpha\}_{\alpha\in\Lambda} \subset K(\lambda,\eta)$ with the following property:

$$\left\langle x, Ey^*\right\rangle = \lim_{\alpha\in\Lambda}\left\langle x, \Upsilon(g_\alpha) y^*\right\rangle = \\ = \lim_{\alpha\in\Lambda}\left\langle x, \left(1 - \Upsilon(1 - f_\alpha)\right) y^*\right\rangle$$

for all $x \in X$. Since $\left\langle x, Ey^*\right\rangle = \left\langle x, y^*\right\rangle$ we have $y^* \in Rang(E)$.

For any $\theta > 0$, there is $\eta_0 > 0$ such that $0 \le f(t) \le \frac{\theta}{2}$ for all $t \in [\lambda, \lambda + \eta_0]$, so for $E \in K(\lambda, \eta_0)$ there is a net $\{g_\alpha\}_{\alpha\in\Lambda} \subset F(\lambda,\eta_0)$ with the property:

$$weak - \lim_{\alpha\in\Lambda} \Upsilon(g_\alpha) = E\ .$$

Now, we are going to apply the fourth condition of the definition:

$$\int_{[a,b]} \left|(fg_\alpha)'\right| = \int_{[a,b]} \left|f' g_\alpha + f\ g_\alpha'\right| \le \\ \le \int_{[\lambda,\lambda+\eta_0]} \left|f' g_\alpha\right| + \int_{[\lambda,\lambda+\eta_0]} \left|+f\ g_\alpha'\right| \le \\ \le \frac{\theta}{2} + \frac{\theta}{2} = \theta,$$

so:

$$\left|\left\langle x, \Upsilon(f) x^*\right\rangle\right| \le \left|\left\langle x, \Upsilon(f) y^*\right\rangle\right| = \\ = \left|\left\langle x, \Upsilon(f) Ey^*\right\rangle\right| = \\ = \left|\left\langle \Phi(f) x, Ey^*\right\rangle\right| = \\ = \left|\lim_{\alpha\in\Lambda}\left\langle \Phi(f) x, \Upsilon(g_\alpha) y^*\right\rangle\right| = \\ = \left|\lim_{\alpha\in\Lambda}\left\langle x, \Upsilon(f\ g_\alpha) y^*\right\rangle\right| \le \\ \le \sup_{\alpha\in\Lambda}\left\|\Upsilon(f\ g_\alpha)\right\| \left\|x\right\| \left\|y^*\right\|$$

for all $y^* \in X^*$so $E \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$. Thus, from the inequality $\left|\left\langle \Upsilon(f)x, y^*\right\rangle\right| \le \theta\|\Upsilon\|\|x\|\|y^*\|$ follows $\Upsilon(f)x^* = 0$, so the range of E coincides with $Z(\lambda)$; the set E is a projection.

Let us establish that $K(\lambda, \eta)$ is a commutative multiplicative semigroup. Let $\hat{K}, \breve{K} \in K(\lambda,\eta)$, us have that there are nets $\{g_\alpha\}_{\alpha\in\Lambda}, \{h_\beta\}_{\beta\in B} \in F(\lambda,\eta)$ such that:

$$\hat{K} = weak - \lim_{\alpha\in\Lambda} \Upsilon(g_\alpha)$$

and:

$$\breve{K} = weak - \lim_{\beta\in B} \Upsilon(h_\beta).$$

For all $x \in X$, we have:

$$\left\langle x, \hat{K}\ \breve{K}y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\langle x, \Upsilon(g_\alpha)\ \breve{K}y^*\right\rangle = \\ = \lim_{\alpha\in\Lambda}\left\langle \left(\Upsilon(g_\alpha)\right)^* x, \breve{K}y^*\right\rangle = \\ = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \left(\Upsilon(g_\alpha)\right)^* x, \Upsilon(h_\beta) y^*\right\rangle\right\} = \\ = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle x, \Upsilon(g_\alpha h_\beta) y^*\right\rangle\right\} = \\ = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle x, \Upsilon(h_\beta)\Upsilon(g_\alpha) y^*\right\rangle\right\} = \\ = \lim_{\alpha\in\Lambda}\left\langle x, \breve{K}\Upsilon(g_\alpha)\ y^*\right\rangle = \\ = \lim_{\alpha\in\Lambda}\left\langle \left(\breve{K}\right)^* x, \Upsilon(g_\alpha) y^*\right\rangle = \\ = \left\langle \left(\breve{K}\right)^* x, \hat{K}y^*\right\rangle = \left\langle x, \breve{K}\hat{K}\ y^*\right\rangle,$$

so $\widehat{K}\,\breve{K} = \breve{K}\widehat{K}$, thus:

$$E(\lambda) \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta),$$

uniqueness is follows from the properties of the projections. We define the set of the projection $\{E(\lambda)\}_{\lambda\in[a,b]}$ on X by presuming $E(\lambda) = O$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda > b$.

Now, let us establish the properties $\{E(\lambda)\}_{\lambda\in[a,b]}$. Assuming that $a \le \lambda < \mu < b$, and assuming η is large enough, we are going to obtain that from $E(\lambda)$, $E(\mu) \in K(\lambda, \eta)$ follows $E(\lambda)$, $E(\mu) = E(\mu)\, E(\lambda) = E(\lambda)$. If $\eta = \mu - \lambda$, then from $E(\lambda) \in K(\lambda, \eta)$ follows the existence of the nets $\{g_\alpha\}_{\alpha\in\Lambda} \in F(\lambda,\eta)$ and $\{h_\beta\}_{\beta\in\mathrm{B}} \in F(\lambda,\eta)$ with the properties:

$$weak - \lim_{\alpha\in\Lambda} \Upsilon(g_\alpha) = E(\lambda)$$

and:

$$weak - \lim_{\beta\in\mathrm{B}} \Upsilon(h_\beta) = E(\mu).$$

Next, since $g_\alpha h_\beta = g_\alpha$ we have:

$$\begin{aligned}
&\langle x, E(\lambda)\,E(\mu)y^*\rangle = \\
&= \lim_{\alpha\in\Lambda}\langle x, \Upsilon(g_\alpha)\,E(\mu)y^*\rangle = \\
&= \lim_{\alpha\in\Lambda}\langle \Phi(g_\alpha)x, E(\mu)\,y^*\rangle = \\
&= \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in\mathrm{B}}\langle \Phi(g_\alpha)x, \Upsilon(h_\beta)\,y^*\rangle\right\} = \\
&= \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in\mathrm{B}}\langle \Phi(h_\beta)\Phi(g_\alpha)x, y^*\rangle\right\} = \\
&= \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in\mathrm{B}}\langle \Phi(g_\alpha)x, y^*\rangle\right\}
\end{aligned}$$

for all $x \in X$, $y^* \in X^*$. So, we have obtained $\langle x, E(\lambda)\,E(\mu)y^*\rangle = \langle x, E(\lambda)\,y^*\rangle$ and thus equality:

$$E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$$

holds for all $a \le \lambda < \mu < b$.

Since:

$$strong - \lim_{\mu\to\lambda+0} E(\mu) = E(\lambda+0)$$

we have $E(\lambda + 0) \in K(\lambda)$.

For any pair $x \in X$, $y^* \in X^*$ and any function $f \in AC([a, b])$ the morphism $f \mapsto \langle x, \Upsilon(f)y^*\rangle$ is an element of the dual space to $AC([a, b])$ and since $AC([a, b])$ is isometric to $L^1([a,b]) \oplus C$, from the duality argument, we have that there are $\gamma\langle x, y^*\rangle \in L^\infty([a,b])\tilde{c}\langle x, y^*\rangle \in C$, which satisfy the following equality:

$$\begin{aligned}
&\langle x, \Upsilon(f)y^*\rangle = \tilde{c}\langle x, y^*\rangle f(b) + \\
&+ \int_{[a,b]} f'(t)\gamma\langle x, y^*\rangle(t)dt.
\end{aligned}$$

For any $\lambda \in [a,b)$, we assume $0<\lambda+\eta<b$ then the function:

$$g(\lambda,\eta)(t) = \begin{cases} 1 \quad \text{on} \quad [a,\lambda] \\ \text{ecreasing} \quad on \quad [\lambda, \lambda+\eta] \\ 0 \quad on \quad [\lambda+\eta, b] \end{cases}$$

belongs to $F(\lambda,\eta)$ and:

$$\langle x, \Upsilon(g(\lambda,\eta))y^*\rangle = -\frac{1}{\eta}\int_{[\lambda,\lambda+\eta]} \gamma\langle x, y^*\rangle(t)dt\cdot$$

Thus, there is a weak limit $g(\lambda,\eta) \xrightarrow{weak-\eta\to 0+} E(\lambda)$.

So, λ-almost everywhere, we obtain $\gamma\langle x, y^*\rangle(\lambda) = -\langle x, E(\lambda)y^*\rangle$ and for arbitrary $x\in X$, $y^* \in X^*$, the integral equality:

$$\begin{aligned}
&\langle x, \Upsilon(f)y^*\rangle = \langle x, y^*\rangle f(b) - \\
&- \int_{[a,b]} f'(\lambda)\langle x, E(\lambda)y^*\rangle d\lambda
\end{aligned}$$

holds for all $f \in AC([a,b])$. Thus, by taking $f(\lambda) = \lambda$, we have:

$$\begin{aligned}
&\langle Ax, y^*\rangle = \langle x, A^*y^*\rangle = \\
&= b\langle x, y^*\rangle - \int_{[a,b]}\langle x, E(\lambda)y^*\rangle d\lambda.
\end{aligned}$$

Let function $\varphi \in L^1([a,b])$ then we can define:

$$f(\varphi) = \int_{[\lambda,b]} \varphi(t)dt$$

thus $f(\varphi) \in AC([a,b])$ and almost everywhere $f'(\varphi)(\lambda) = -\varphi(\lambda)$. For any fixed $x \in X$, the mapping $A(x)(\varphi) = \Phi(f(\varphi))(x)$ is continuous as the mapping $L^1([a,b]) \to X$. So, we have:

$$\begin{aligned}
&\langle A(\varphi)x, y^*\rangle = \langle \Phi(f(\varphi))x, y^*\rangle = \\
&= \int_{[a,b]} \varphi(\lambda)\langle x, E(\lambda)y^*\rangle d\lambda
\end{aligned}$$

and the mapping $A^*(x): X \to L^\infty([a,b])$ is such that:

$$\langle \varphi, A^*(x) y^* \rangle = \int_{[a,b]} \varphi(\lambda) \langle x, E(\lambda) y^* \rangle d\lambda .$$

Theorem 6. Let $\{E(\lambda) \in L(X^*), \quad \lambda \in \mathbb{R}\}$ be a weak spectral family concentrated on [a, b] then there is a linear well-bounded operator $A \in L(X)$ on the reflexive Banach space X such that:

$$\langle A(x), y^* \rangle = b \langle x, y^* \rangle - \int_{[a,b]} \langle x, E(\lambda)\, y^* \rangle d\lambda$$

holds for all $x \in X$, $y^* \in X^*$.

Proof. Assuming $\{E(\lambda) \in L(X^*), \quad \lambda \in \mathbb{R}\}$ is a weak spectral family concentrated on [a, b], the linear operator $A \in L(X)$ can be defined by the following formula:

$$\langle A(x), y^* \rangle = b \langle x, y^* \rangle - \int_{[a,b]} \langle x, E(\lambda) y^* \rangle d\lambda ,$$

it is easy to see that this operator is linear and the only property of it that has to be established is well-boundedness.

By the induction and the Fubini theorem, we have:

$$\begin{aligned} \langle (A(x))^n, y^* \rangle &= b^n \langle x, y^* \rangle - \\ &- \int_{[a,b]} n \lambda^{n-1} \langle x, E(\lambda) y^* \rangle d\lambda, \end{aligned}$$

thus:

$$\left\| (A(x))^n \right\| \le b^n + n \sup_{\lambda \in [a,b]} \{ \| E(\lambda) \| \} \int_{[a,b]} \lambda^{n-1} d\lambda$$

and operator A is well-bounded.

Continuous Functional Calculus on Lebesgue Spaces

Theorem 7. Let A be a well-bounded linear operator on Lebesgue spaces $L^p(\Omega, \Sigma, \mu), p \in (1, \infty)$. Then the operator A is a scalar type spectral operator.

Proof. The spectral family $\{E(\lambda)\}$ of operator A is concentrated on the interval $[a,b] \subset \mathbb{R}$.

Let us assume that $u \in L^p(\Omega, \Sigma, \mu), p \in (1, \infty)$ and $v \in L^q(\Omega, \Sigma, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We have to show that the variation of the function $\langle E(\lambda) u, v \rangle$ is bounded as the function of λ. Assume that $a = \lambda_0 < \lambda_1 < \ldots < \lambda_n = b$ is a partition of the interval [a, b]. For arbitrary elements $u \in L^p(\Omega, \Sigma, \mu), p \in (1, \infty)$ and $v \in L^q(\Omega, \Sigma, \mu)$, the variation of the function $\langle E(\lambda) u, v \rangle$ equals:

$$\begin{aligned} &\operatorname*{var}_{[a,b]} \langle E(\lambda) u, v \rangle = \\ &= \sum_{i=1,\ldots,n} \left| \langle E(\lambda_i) u, v \rangle - \langle E(\lambda_{i-1}) u, v \rangle \right| = \\ &= \sum_{i=1,\ldots,n} \left| \langle (E(\lambda_i) - E(\lambda_{i-1})) u, v \rangle \right| \le \\ &\le \left\| \sum_{i=1,\ldots,n} (E(\lambda_i) - E(\lambda_{i-1})) \right\| \|u\| \|v\|. \end{aligned}$$

Let m be an integer such that $\lambda_{m-1} < c < \lambda_m$, so we have:

$$\begin{aligned} &\left\| \sum_{i=1,\ldots,n} (E(\lambda_i) - E(\lambda_{i-1})) \right\| \le \\ &\le \left\| \sum_{i=1,\ldots,m-1} (E(\lambda_i) - E(\lambda_{i-1})) \right\| + \\ &+ \left\| (E(\lambda_m) - E(\lambda_{m-1})) \right\| + \\ &+ \left\| \sum_{i=m+1,\ldots,n} (E(\lambda_i) - E(\lambda_{i-1})) \right\| \end{aligned}$$

thus for $\lambda < c$ we have $\|E(\lambda)\| \le 1$, and for $\lambda \ge c$ we have $\|I - E(\lambda)\| \le 1$. So $\|E(\lambda_m)\| \le 2$ and $\|E(\lambda_{m-1})\| \le 1$. Since $\{E(\lambda_i)\}_{i=1,\ldots,m-1}$ and $\{I - E(\lambda_{n-i})\}_{i=m-1,\ldots,n}$ are the increasing sequences of contractive projections, we have:

$$\left\| \sum_{i=1,\ldots,m-1} (E(\lambda_i) - E(\lambda_{i-1})) \right\| \le 2(q-1)$$

and:

$$\left\| \sum_{i=m-1,\ldots,n} (E(\lambda_i) - E(\lambda_{i-1})) \right\| \le 2(q-1).$$

In the final conclusion, we obtain:

$$\left\| \sum_{i=1,\ldots,n} (E(\lambda_i) - E(\lambda_{i-1})) \right\| \le 4(q-1) + 3 ,$$

thus, the variation of $\langle E(\lambda) u, v \rangle$ cannot exceed the value $(4(q-1)+3)\|u\|\|v\|$. The theorem is proven.

Definition 5. A solitary operator is a bounded linear surjective operator $U : X \to X$ on a Banach space that for all $x \in X$ and $y \in X^*$ satisfies the following equality:

$$\langle Ux, U^* y \rangle = \langle x, y \rangle$$

where, $U^* : X^* \to X^*$.

Theorem 8. Assuming (Φ, *X*) is a functional calculus on the measurable space (Z, Σ). Then there are a semi-finite measure space (Ω, *F*, μ) and solitary operator $U : X \to L^p(\Omega, \mathsf{F}, \mu)$ and an injective pointwise continuous $*$ -homomorphism $F : M(Z, \Sigma) \to M(\Omega, \mathsf{F})$, such that $\Phi(f) = UM_{Ff}U^{-1}$, where M_{Ff} is the operator of the multiplication by *f*.

Proof. For every set $A \in \Sigma$, we define measure $\mu_x(A) = \langle \Phi(\chi_A)x, x^* \rangle$ as a function of $x \in X$, so $\langle \Phi(f)x, x^* \rangle = \langle \Phi(f) \rangle_{\mu_x}$ for every bounded *f*. Now, for every bounded *f*, we define the space $B_x = \left[\{\Phi(f)x,\ f \in M_b(Z, \Sigma)\}\right]$, thus there is a solitary operator $W_x : L^p(Z, \Sigma, \mu_x) \to B_x$ as an extension of mappings $M_b(Z, \Sigma) \to B_x$ and $f \to \Phi(f)x$.

Let $\{x_i\}$ and $\{x_i^*\}$ be two sets of unit vectors in *X* and X^* spaces, respectively, with properties:

$$\langle x_k, x_k^* \rangle = \|x_k\| \|x_k^*\|_* = 1 \forall k \in N$$

and:

$$\langle x_i, x_k^* \rangle = 0$$

for every $i \neq k$.

For every *k*, we can define the set $Z_k = Z \times \{k\}$ as an exemplar of *Z* then the set Ω can be represented as the disjoint union $\bigcup_k Z_k$. Let

Let us define an additive set function μ by the following formula:

$$\mu(A) = \sum_k \mu_{x_k}(A \cap Z_k) \quad \forall A \in \mathsf{F}.$$

The additive set function μ is the measure on the maximal sigma-algebra *F* on Ω, which includes all measurable mapping $Z_k = Z \times \{k\}$ into Ω.

The operator W_{x_k} is correctly defined on $L^p(Z_k, \Sigma, \mu_{x_k})$ and $W_{x_k} : L^p(Z_{x_k}, \Sigma, \mu_{x_k}) \to B_{x_k}$ so we define the operator $U : X \to L^p(\Omega, \mathsf{F}, \mu)$ by the condition $U^{-1} = W_{x_k}$ on $L^p(Z_k, \Sigma, \mu_{x_k}) \subseteq L^p(\Omega, \mathsf{F}, \mu)$.

Then the $*$-homomorphism $F : M(Z, \Sigma) \to M(\Omega, \mathsf{F})$, we introduce by the formula:

$$(F\ f)(x, k) = f(x), \quad x \in X$$

For all $f \in (Z, \Sigma)$, we define the multiplication operator calculus as $M_{Ff} = U\Phi(f)U^{-1}$, so the theorem has been proven.

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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