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On the Variational Symmetries of P.D.E's Incorporating Boundary Value Constraints

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Abstract: A crucial interface between Optimization theory, Trace theory, and Lie Symmetry theory is brought to the fore in this paper, as an enhancement of standard known results established in Partial Differential Equation (P.D.E) analysis. In particular, the incorporation of compatible Boundary Value constraints in the re-assessment of admitted variational symmetries is a key process in the development of the discussions and results. Classical variational formulation techniques for a few Boundary Value Problems are considered. Ramifications of standard and modified admitted variational symmetries in simplification of PDEs constitute further relevant crucial details addressed appreciably.

Keywords: Boundary Value Problems, Calculus of Variations, Classical Lagrangians, Symmetry Invariant Solutions, Laplace's Equation, Poisson Equation

Introduction

The utility of admitted variational symmetries in the simplification of Differential Equations should be reckoned with promptly: they are instrumental in reducing the order of such equations by two (Springer Nature, 1990). Reckoning also that a wide variety of P.D.E's with applications to industry and the physical world are formulated from the calculus of variations, the importance of considerations stemming from this vantage point becomes more evident and compelling. For a sufficiently regular functional $E = \int_{\Omega} F(x, v, \nabla v) d\mu$ where by Ω is a pseudo-Riemannian submanifold of $\mathbb{R}^n$, $x = (x_i)_{i=1}^{n}$ represents the coordinates on Ω and $v(x)$ is a varying functional acting on Ω. Perturbation of x and v via the calculus of variations enables us to formulate the multivariate Euler-Lagrange equations: A P.D.E or system of P.D.E's that permits us to compute those functions $\bar{v}$ which fit in with optimality conditions of E (Opara, 2020). These equations are given by:

$$F_v(x, \bar{v}, \nabla\bar{v}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{v_{xi}}(x, \bar{v}, \nabla\bar{v})$$

An alternative equivalent expression for the Euler-Lagrange equations is provided by Jost and Li-Jost (1998). There are also optimization theorems that guarantee the existence and/or uniqueness of the optimizing function $\bar{v}$, but a closer look into cases where Ω has a non-trivial topological boundary is hereby demanded. This is because, starting with the Euler-Lagrange equations for the converse and formulating the functional E via the Lax-Milgram theorem and Green's theorems, we typically do not obtain the identical functional E in the former case: there is usually a trace boundary term realized in addition after inputting the Euler-Lagrange equations in the Lax-Milgram formulation procedure. We shall address this observation in considerable detail with the aid of some classical tools from Lie Group theory and Sobolev Space theory.

Analysis of Variational Symmetries Compatible with Boundary Value Constraints in Well-Posed P.D.E

A pair of simple and common elliptic Boundary Value Problems (B.V.P) shall be used for this study, exploiting hindsight of knowledge of classical fundamental solutions in building current key results and propositions. For the first B.V.P, we take Laplace's equation on $\mathbb{R}^2$ with the following specified constraints:

$$(P_1)\begin{cases}\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0\ \ in\ B(0,2)\\ u(x,y)|_{\partial B(0,2)}=ln8\\ \nabla u(x,y)|_{\partial B(0,2)}=\dfrac{3(x,y)}{4}\end{cases}$$

The solution to (P_1) is $u(x,y)=\frac{3}{2}ln(x^2+y^2)$, which is compatible with the infinitesimal symmetry $y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$: a generator of the rotation group on the Riemannian manifold $B(0,2)\subset\mathbb{R}^2$. This infinitesimal generator is also a variational symmetry of Laplace's equation in the usual sense, without the imposition of boundary constraints.

For the second B.V.P, we take Poisson's equation, as given below:

$$(P_2)\begin{cases}\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=f\ \ in\ \Omega\\ u(x,y)|_{\partial\Omega}=0\end{cases}$$

The set Ω above is a bounded open subset of $\mathbb{R}^2$ with a C^1 topological boundary, and f is a fixed function in the Hilbert Space $L^2(\Omega)$. The solution to (P_2) may be compatible with some infinitesimal symmetry of Laplace's equation: $\xi\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}$, given that f satisfies the first-order linear P.D.E :

$$\xi\frac{\partial f}{\partial x}+\tau\frac{\partial f}{\partial y}=\left(\frac{\partial\eta}{\partial u}-\frac{\partial\xi}{\partial x}-\frac{\partial\tau}{\partial y}\right).f\qquad (P_3)$$

The said infinitesimal generator is a (pseudo-) variational symmetry of Poisson's equation (P_2) given that η is a constant multiplied by u.

The P.D.E's (without boundary constraints) in (P_1) and (P_2) are obtained as Euler-Lagrange equations in the process of optimizing specific `Energy Functionals' expressed as integrals of Lagrangians. When these Euler-Lagrange equations are implemented in formulations from the Lax-Milgram theorem and Green's theorems, then additional (trace) terms emerge to modify the original Energy functions. We shall refer to these modified functionals as the 'Total Energy Functionals'. It is a worthwhile venture to identify how the boundary trace terms in these modified functionals are extensions from interiors of the manifolds of the definition of the B.V.P's in such a way as to maintain the admittance of variational symmetries by the modified functionals.

Materials and Methods

Computational Procedures for Confirmation of Variational Symmetries

For Laplace's equation (P_1), the infinitesimal rotation group generator $y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$ is admitted by the equation $\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\right]$, that is, (P_1) without boundary constraints. Computing a group invariant solution with this observation, we obtain $u(x,y)=k.ln(x^2+y^2)$. The manifold chosen in this setting is the ball $B(0,r)$ due to the convenience of evaluating the group invariant solution, as confined to its boundary. For $k=\frac{3}{2}$ and $r=2$, the B.V.P (P_1) is realized (although generalization to arbitrary positive constants k and r would equally corroborate findings). We shall show computationally that the same rotational symmetry known to be admitted without boundary constraints is admitted as a variational symmetry by the B.V.P (P_1), by implementing the standard vector prolongation technique and Stokes' Theorem. Before commencing computations, we reckon that the weak formulation of this equation is carried out in the Hilbert space $H^1(B(0,2))$. We need to make reference to the classical Lax-Milgram theorem stated below to formulate the optimized functional required, as the integral of a Lagrangian.

Lax-Milgram theorem: (Brezis, 2011.) Assume that $a(u,v)$ is a continuous and coercive bilinear form on a Hilbert Space H. Then given any $\varphi\in H^*$, there exists a unique $u\in H$ such that:

$$a(u,v)=\langle\varphi,v\rangle\ \ \forall v\in H.$$

Moreover, if a is symmetric, then u is characterized by:

$$\tfrac{1}{2}a(u,u)-\langle\varphi,u\rangle=\min_{v\in H}\left\{\tfrac{1}{2}a(v,v)-\langle\varphi,v\rangle\right\}$$

Now, $\forall v\in H^1(B(0,2))$, we have the following:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\ \Rightarrow\ \int_{B(0,2)}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).v=0$$
$$\Rightarrow\begin{cases}\displaystyle\int_{B(0,2)}\left(\frac{\partial u}{\partial x}.\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}.\frac{\partial v}{\partial y}\right)\\ \displaystyle-\int_{\partial B(0,2)}\frac{\partial h}{\partial N}.vd\sigma=0\end{cases}$$

The final line above is determined from Green's formula for multivariate integration. The function h is the restriction of u to $\partial B(0,2)$, N is the Gauss map on $\partial B(0,2)$, and $\frac{\partial h}{\partial N}=\langle\nabla h,N\rangle$. The surface element $d\sigma$ is obtained as:

$$d\sigma=n_1dy-n_2dx,\quad where\ N=(n_1,n_2)$$

Applying the Lax-Milgram theorem with the symmetric, continuous, and coercive bilinear form $a(u,v)=\int_{B(0,2)}\nabla u.\nabla v$ on $H^1(B(0,2))$; and the functional φ in the dual of $H^1(B(0,2))$ given by:

$$\varphi(v) = \int_{\partial B(0,2)} \frac{\partial h}{\partial N}.vd\sigma\,,$$

we establish the unique existence of a solution u to (P_1) in $H^1(B(0,2))$. We also importantly establish that:

$$u = \min_{v\in H}\left\{\tfrac{1}{2}\int_{B(0,2)} \|v\|^2 - \int_{\partial B(0,2)} \frac{\partial h}{\partial N}.vd\sigma\right\}.$$

We now have to express the total energy functional within curly brackets in the above optimization problem under one common integral over the Riemannian manifold $B(0,2)$ with the aid of Stokes' theorem. For (P_1), $d\,\sigma$ is easily computed, as the Gauss map (outward unit normal) here is simply $N = \left(\frac{x}{2}, \frac{y}{2}\right)$. Hence, the differential one-form $\omega = \frac{\partial h}{\partial N}.vd\sigma$ is computed:

$$\omega = \langle \nabla h, N\rangle\, vd\sigma = \frac{3}{8}(x^2+y^2).v.\left(\frac{x}{2}dy - \frac{y}{2}dx\right).$$

Its exterior derivative $d\omega$ is thereby computed as:

$$d\omega = \frac{3}{16}\left[4v(x^2+y^2)dx\wedge dy + (x^2+y^2)(xv_x+yv_y)dx \wedge dy\right].$$

From Stokes' theorem, we have $\int_{\partial B(0,2)}\omega = \int_{B(0,2)} d\omega$, giving us a total energy functional here to be:

$$\int_{B(0,2)}\left(\frac{1}{2}\|v\|^2 - \frac{3}{16}\left[4v(x^2+y^2) + (x^2+y^2)(xv_x+yv_y)\right]\right)dx\wedge dy \longrightarrow (E_1).$$

Assume that (E_1) admits a variational symmetry $\mathbf{v} = \left[\xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u}\right]$.

The infinitesimal variational symmetry criterion is hereby given as:

$$pr^{(1)}\mathbf{v}[L] + L.div(\psi) = 0 \iff \left(\xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^y\frac{\partial}{\partial u_y}\right)[L] + L.\left(D_x\xi + D_y\tau\right) = 0\,,$$

where L is the Lagrangian of (E_1) with v replaced by u, $\psi = (\xi\,,\tau)$ and, $\eta^x = D_x\eta - u_xD_x\xi - u_yD_x\tau$, $\eta^y = D_y\eta - u_xD_y\xi - u_yD_y\tau$.

For details on how the infinitesimal symmetry criterion is obtained, we refer the reader to (Springer Nature, 1990). Developing the formulation of the infinitesimal symmetry criterion for this case, we have:

$$\begin{aligned}&\left(\xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^y\frac{\partial}{\partial u_y}\right)[L] + L.\left(D_x\xi + D_y\tau\right) = 0 \iff \\ &\xi\left(-\tfrac{3}{2}u.x - \tfrac{3}{16}u_x(x^2+y^2) - \tfrac{3}{8}x(x.u_x + y.u_y)\right) + \tau\left(-\tfrac{3}{2}u.y - \tfrac{3}{16}u_y(x^2+y^2) - \tfrac{3}{8}y(x.u_x+y.u_y)\right) + \eta\left(-\tfrac{3}{4}(x^2+y^2)\right) \\ &+ \left(D_x\eta - u_xD_x\xi - u_yD_x\tau\right)\left(u_x - \tfrac{3}{16}x(x^2+y^2)\right) \\ &+ \left(D_y\eta - u_xD_y\xi - u_yD_y\tau\right)\left(u_y - \tfrac{3}{16}y(x^2+y^2)\right) \\ &+ \left(\frac{1}{2}{u_x}^2 + \frac{1}{2}{u_y}^2 - \frac{3}{16}\left[4u(x^2+y^2) + (x^2+y^2)(xu_x+yu_y)\right]\right)\left(D_x\xi + D_y\tau\right) = 0\end{aligned}$$

For the infinitesimal symmetry criterion to be met as expanded above, we must have the coefficients of $\{u_x, u_y,\ \text{their powers, and products of}\}$ to be zero. Moreover, the sum of all terms free of u_x and u_y must also equal zero. We thereby determine the following table to evaluate the coefficients of these identified monomials. (Note that $D_x\xi = \xi_x + \xi_u u_x$, $D_y\xi = \xi_y + \xi_u u_y$ and so on.)

Monomials	Coefficients
Free	$-\frac{3}{2}ux\xi - \frac{3}{2}uy\tau - \frac{3}{4}x^2\eta - \frac{3}{4}y^2\eta - \frac{3}{16}x^3\eta_x - \frac{3}{16}xy^2\eta_x - \frac{3}{16}yx^2\eta_y - \frac{3}{16}y^3\eta_y - \frac{3}{4}ux^2\xi_x - \frac{3}{4}uy^2\xi_x - \frac{3}{4}ux^2\tau_y - \frac{3}{4}uy^2\tau_y$
${u_x}^2$	$\eta_u - \frac{1}{2}\xi_x + \frac{1}{2}\tau_y$
${u_x}^3$	$-\frac{1}{2}\xi_u$
$u_x.\,u_y$	$-\tau_x - \xi_y$
$u_x.{u_y}^2$	$-\frac{1}{2}\xi_u$
${u_x}^2.u_y$	$-\frac{1}{2}\tau_u$
${u_y}^2$	$\eta_u + \frac{1}{2}\xi_x - \frac{1}{2}\tau_y$
${u_y}^3$	$-\frac{1}{2}\tau_u$

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u_x	$-\frac{9}{16}x^2\xi-\frac{3}{16}y^2\xi-\frac{3}{8}xy\tau+\eta_x-\frac{3}{16}\eta_u(x^3+xy^2)+\frac{3}{16}\xi_y(y^3+yx^2)-\frac{3}{16}\tau_y(x^3+xy^2)$
u_y	$-\frac{9}{16}y^2\tau-\frac{3}{16}x^2\tau-\frac{3}{8}xy\xi+\eta_y-\frac{3}{16}\eta_u(y^3+yx^2)+\frac{3}{16}\tau_x(x^3+xy^2)-\frac{3}{16}\xi_x(y^3+yx^2)$

Results

Upon equating the coefficients from the above table to zero, we can conclude with relative ease that the infinitesimal variational symmetry criterion for (P_1) is met if and only if ξ = y, τ = -x, η = 0. This is precisely the rotational symmetry that was identified as being compatible with the solution to (P_1). This is an exemplary computation with a result that may be suitably generalized to a similar B.V.P of Laplace's equation, as pointed out earlier. As a noteworthy remark, (P_1) without boundary value constraints is famously the Euler-Lagrange equation for optimizing the Dirichlet Energy functional, that is:

$$\frac{1}{2}\int_{\Omega}\|v\|^2=\frac{1}{2}\int_{\Omega}\left({v_x}^2+{v_y}^2\right)\ :v\in H^1(\Omega)$$

On the other hand, beginning with the Euler-Lagrange equation to formulate the functional which it optimizes brings up an additional boundary term via Green's theorem (analogous to a constant of integration in single variable integration) which can be processed further via Stokes' theorem for the purpose of performing the requisite vector field prolongation. It is this modification of the Dirichlet Energy we hereby refer to as the Total Energy Functional. One may desire to view the outcome of obtaining the Euler-Lagrange equations from this functional in (E_1), as done below:

$$L_v(x,\bar{v},\nabla\bar{v})=\sum_{i=1}^{2}\frac{\partial}{\partial x_i}L_{v_{xi}}(x,\bar{v},\nabla\bar{v})$$
$$\Rightarrow -\tfrac{3}{4}(x^2+y^2)=\frac{\partial}{\partial x}\left[v_x-\tfrac{3}{16}x(x^2+y^2)\right]+\frac{\partial}{\partial y}\left[v_y-\tfrac{3}{16}y(x^2+y^2)\right]$$
$$\Rightarrow -\tfrac{3}{4}(x^2+y^2)=v_{xx}-\tfrac{9x^2}{16}-\tfrac{3y^2}{16}+v_{yy}-\tfrac{9y^2}{16}-\tfrac{3x^2}{16}$$
$$\Rightarrow -\tfrac{3}{4}(x^2+y^2)=v_{xx}+v_{yy}-\tfrac{12}{16}(x^2+y^2)$$
$$\Rightarrow v_{xx}+v_{yy}=0$$

Clearly, accurate incorporation of the trace boundary portion in the Total Energy Functional for this case does not alter the original Euler-Lagrange equation, in view of the strict variational symmetry present in this instance. A more elaborate description of the concept of alteration of Euler-Lagrange equations obtained from total energy functionals in cases of strict and pseudo-variational symmetries would enhance this practical theory quite richly. In what ensues, analysis of the second referenced B.V.P (P_2) introduces a proposed prospect for further development of this concept.

For Poisson's equation (P_2), let $\psi(x,y)$ be a harmonic function, that is, a solution to (P_1) without its boundary constraints. The generic infinitesimal group generator:

$$\mathrm{v}=k_j\left[\alpha_j(x,y)\frac{\partial}{\partial x}+\beta_j(x,y)\frac{\partial}{\partial y}\right]+\left(ku+\psi(x,y)\right)\frac{\partial}{\partial u}$$
$$:=\ \xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}$$

is admitted by the equation $\left[\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=f\right]$ in (P_2), where $\{k_j\}$'s and k are arbitrary real constants and $\{\alpha_j(x,y)+i.\beta_j(x,y)\}$ constitute the collection of analytic complex-valued functions ($i=\sqrt{-1}$), given that f satisfies the first order linear P.D.E (P_3):

$$\xi\frac{\partial f}{\partial x}+\tau\frac{\partial f}{\partial y}=\left(\frac{\partial\eta}{\partial u}-\frac{\partial\xi}{\partial x}-\frac{\partial\tau}{\partial y}\right).f$$

This observation is made by direct application of the infinitesimal symmetry criterion (Springer Nature, 1990):

$$pr^{(1)}\mathrm{v}\left[u_{xx}+u_{yy}-f\right]=0\quad\text{whenever}\quad u_{xx}+u_{yy}-f=0$$

for any admissible infinitesimal generator $\mathrm{v}=\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}$. The Lie Algebras spanned by $\psi(x,y)\frac{\partial}{\partial u}$ and $\left[\alpha_j(x,y)\frac{\partial}{\partial x}+\beta_j(x,y)\frac{\partial}{\partial y}\right]$ are identified as infinite-dimensional sub-algebras of the overall admitted infinitesimal symmetry.

Poisson's equation is determined as the Euler-Lagrange equation for optimizing the functional:

$$\int_{\Omega}\left(\frac{1}{2}\|v\|^2-v.f\right)dx\wedge dy:=\int_{\Omega}L\,dxdy\ \longrightarrow(E_2)$$

If in addition $\psi(x,y)=0$ above, then the aforementioned symmetry admitted by Poisson's equation is a pseudo-variational symmetry of the Lagrangian in (E_2), in the sense that:

$$pr^{(1)}v[L]+L.div(\xi,\tau)=2k.L$$

for this sub-case. In other words, those one-parameter infinitesimal generators combining just the vectors $\frac{\partial}{\partial x}$

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and $\frac{\partial}{\partial y}$ are the strict variational symmetries of variational problem (E_2). $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are identified with the unit canonical basis vectors in the axial directions of x and y respectively.

For a well-posed system with (P_2), we introduce a second boundary value constraint for the gradient of the dependent variable u. We hereby fix the function f here to be $f(x,y) = ln(x^2+y^2)$, to fit in with the admissibility of symmetries identified above, while we maintain the manifold of definition to be $\Omega = B(0,2)$ as in (P_1) for convenience of computations. Hence, we set:

$$\nabla u(x,y)|_{\partial\Omega} = (ln2-1)(x,y)$$

and with these additional specifications, (P_2) has a solution:

$$u(x,y) = \tfrac{1}{4}(x^2+y^2-4)(ln(x^2+y^2)-2),$$

which is compatible with the same infinitesimal rotation group generator: $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ as seen in (P_1).

Now, beginning with the Euler-Lagrange equation for (E_2) to weakly formulate the total energy functional which it optimizes, for all $v \in H_0^1(B(0,2))$, we have the following:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \quad \Rightarrow \quad \int_{B(0,2)} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f\right).v = 0$$

$$\Rightarrow \quad \begin{cases} -\int_{B(0,2)} \left(\frac{\partial u}{\partial x}.\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial y}\right) - \int_{B(0,2)} f.v \\ \qquad + \int_{\partial B(0,2)} \frac{\partial h}{\partial N}.vd\sigma \;\; = 0 \end{cases}$$

In the final line above due to Green's theorem, h is the restriction of u to $\partial B(0,2)$. Via engagement of the Lax-Milgram theorem, we deduce that u is characterized by:

$$u = \min_{v \in H_0^1(\Omega)} \left\{ \frac{1}{2}\int_\Omega \|v\|^2 - \int_{\partial\Omega} \frac{\partial h}{\partial N}.vd\sigma + \int_\Omega f.v \right\}$$

Recalling the identity: $\frac{\partial h}{\partial N} = \langle \nabla h, N \rangle$ with the Gauss map $N = \left(\frac{x}{2}, \frac{y}{2}\right)$ on $\partial\Omega$ and implementing Stokes' theorem as in (P_1), we arrive at the total energy functional for this case:

$$\int_{B(0,2)} \left(\frac{1}{2}\|v\|^2 - \frac{1}{4}(ln2-1)(x^2+y^2)\big(4v + xv_x + yv_y\big) \right. \\ \left. + \; v.ln(x^2+y^2)\right) dx \wedge dy$$

We may engage the infinitesimal symmetry criterion for variational symmetries as done previously for (P_1) to detect that $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ is again a (strict) variational symmetry for this total energy functional. Setting $[k = ln2-1]$ here, the Euler-Lagrange equation from this total energy functional yields the following:

$$L_v(x,\bar{v},\nabla\bar{v}) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} L_{v_{xi}}(x,\bar{v},\nabla\bar{v})$$

$$\Rightarrow -k(x^2+y^2) + ln(x^2+y^2) = \frac{\partial}{\partial x}\left[v_x - \tfrac{k}{4}(x^2+y^2).x\right] + \frac{\partial}{\partial y}\left[v_y - \tfrac{k}{4}(x^2+y^2).y\right]$$

$$\Rightarrow -k(x^2+y^2) + ln(x^2+y^2) = v_{xx} - \frac{kx^2}{2} - \frac{k}{4}(x^2+y^2) + v_{yy} - \frac{ky^2}{2} - \frac{k}{4}(x^2+y^2)$$

$$\Rightarrow v_{xx} + v_{yy} = \; ln(x^2+y^2)$$

Accurate incorporation of the trace boundary portion in the Total Energy Functional for this case does not alter the original Euler-Lagrange equation, in view of the strict variational symmetry present. The Euler-Lagrange equations for both (P_1) and (P_2) are of the self-adjoint type, which greatly simplifies the concept of determination of all associated Lagrangians hereby considered. Non-self-adjoint differential equations introduce further intricacies in the development of concepts of this sort. Discussions to ensue in the subsequent section present a summarization of motivations and technicalities stemming from the above, including computational illustrations, galvanizing their theoretical and practical utility.

Discussion

Classically, there are three types of P.D.E, namely: elliptic, parabolic, and hyperbolic; of which the prototypical equations are respectively-Laplace's equation, the heat equation, and the wave equation (Braun, 1993). Despite the limitless solutions to these equations prior to the appropriate imposition of boundary value constraints, more emphasis is laid on the `natural' fundamental solutions, which are usually supple to symmetry invariance techniques. Pertaining to Laplace's equation in two independent variables, this P.D.E is of peculiar interest because of the well-known link between its solutions and analytic complex-valued functions. Moreover, as pointed out in an included computational result above, there is a bijective correspondence between admissible symmetries that are in terms of the canonical basis vectors in the axial directions of the two independent variables of Laplace's equation, and the collection of analytic complex-valued functions. Another noteworthy property of Laplace's equation is its admittance of two separate infinite-dimensional Lie sub-algebras as symmetries, with the sub-algebra addressed just above being key in generating invariants for the equation's simplification. Among other prospective concepts, this suggests a viable platform for the study of the special embeddings of $C^\infty(\mathbb{C},\mathbb{C})$ in larger complex Sobolev spaces; as a vantage point for analysis of analytic complex-valued functions, made accessible via standard point symmetries of Laplace's equation. We should note that these symmetries are identical to what is admissible by Poisson's equation, given fulfillment of the earlier identified compatibility first-order P.D.E (P_3); observing that Poisson's equation is the non-homogeneous counterpart of Laplace's equation.

The perhaps redundant consistency of all included computational results above with formally established literature on classical elliptic P.D.E is a necessary precursor to similar brewing concepts stemming into parabolic P.D.E analysis. As a first hint towards this shift, the reliance of symmetry techniques on C^∞ functional formulation is somewhat at odds with the Hilbert spaces of formulation that are demanded in the foundational cited theorems of existence and uniqueness of solution. It is admittedly a thin line to toe in transitioning correctly from tools engaged in the Hilbert Sobolev spaces to those demanded in their dense C^∞ functional subspaces, where the symmetry techniques are implemented. For instance, it is relevant to pay attention to the detail that only almost everywhere correspondence to analytic functional specifications is required for the validity of statements of P.D.E's. With this degree of freedom in mind, the Total Energy functional required to be optimized to generate each Boundary Value Problem may strictly differ from what we find following the implementation of Green's theorem of integration, but only by a Radon measure in any case. This particular difference between the energy functionals is identified as a trivial affine separation, and the broader treatment of other affine separations is a noteworthy detail that could be exploited to expand the scope of computations with pseudo-variational symmetries. Pseudo-variational symmetries include divergence symmetries (Springer Nature, 1990), variational C^∞ symmetries (Muriel *et al*., 2006), and μ −symmetries (Cicogna *et al*., 2004); which are equally as potent in reducing differential equations by the same degree/order known for strict variational symmetries.

It is palpable that a reliable theory linking pseudo-variational symmetries and affine separations between associated Lagrangians can be developed. If so, then in some cases, variational symmetries from alternative Lagrangians of (Total) Energy functionals would be useful for solving a number of B.V.Ps of interest for their profound scientific and didactic applications. This would be of great interest since in the case of dissipative systems (often governed by parabolic PDEs), it would mostly be impossible to derive them from strict variational problems, lest they be linked to classical conservation laws, which is contradictory. Moreover, the involvement of boundary contributions to the Lagrangians being optimized, although not a novel development, is acknowledged as a downplayed aspect of the total package of Noether's theorem (Halder *et al*., 2018); but one which could prove an invaluable piece in validation of extensions to similar techniques for pseudo-variational symmetries.

Conclusion

As another related vital prospective pursuit, the similarities and peculiarities involved in transitioning the weak formulation techniques from elliptic B.V.P to parabolic ones are worth investigating. For instance, there are numerous heat-type equations describing a range of key physical phenomena, and these are all parabolic or weakly parabolic equations. Symmetry invariant solutions tend to characterize equilibrium states of many such physical systems, via possible pseudo-variational formulation techniques tweaked from the classical expositions of this paper. Making this transition from elliptic to parabolic B.V.P's also requires adjusting from the classical Lax-Milgram to accommodate the peculiarities of its parabolic counterpart: J.L. Lion's theorem (Brezis, 2011).

A common ingredient required in the development of all hereby established and suggested principles is a robust grasp of the Trace theory of multivariate integration in Sobolev spaces. It is well-known that the Trace Operator (γ) is linear and continuous in the following map:

$$\gamma: W^{k,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

for all Sobolev spaces $W^{k,p}(\Omega): k \in \mathbb{N} \cup \{0\},\ p \geq 1$. Consider a sufficiently regular first-order Lagrangian functional on $W^{k,p}(\Omega)$ is given by:

$$E(u) = \int_\Omega F(x, u, \nabla u) dV$$

for some open and bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Elements $\bar{u}$ in the kernel of the Frechet differential of E comprise a linear subspace of $W^{k,p}(\Omega)$, such that for any $\bar{u} \in Ker[E']$, we have:

$$\langle E'(\bar{u}), v \rangle = 0 \ \ \forall v \in W^{k,p}(\Omega).$$

For all $u \in W^{k,p}(\Omega)$, $E'(u)$ can be described as a bounded linear functional from the dual space $(W^{k,p}(\Omega))^*$ of $W^{k,p}(\Omega)$. Invariably, the collection of variational-symmetry invariant solutions to the Euler-Lagrange equation(s) associated with some Lagrangian comprises a specific subspace of $Ker[E']$. As a final noteworthy remark, we find that implementation of the infinitesimal vector prolongation technique on just $L^p(\partial\Omega)$ (facilitated by pull-backs into the interior of Ω), for the illustrations considered in this paper, is sufficient to solve for any admitted symmetries. This suggests how systematic engagement of Trace theory could be relevant in symmetry analysis of P.D.E's.

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Author's Contributions

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Ethics

The authors do not foresee any ethical issues that may arise as a result of publication of this manuscript.

Conflict of Interests

The authors declare that there is no conflict of interests with any other party, with regards to this article.

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